A quick introduction to the Einstein equations

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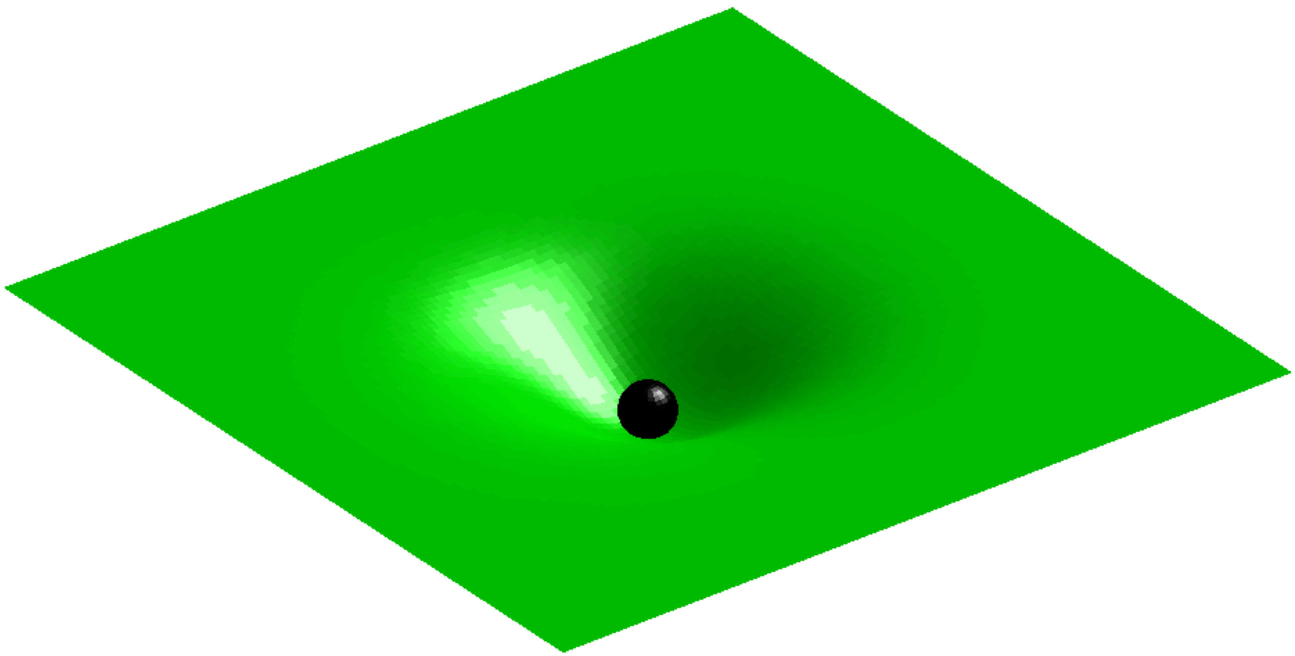


Preview

The Einstein equations are simple geometrical equations to be satisfied by a metric of signature $-+++$ on the 4-manifold representing spacetime. More specifically, they constrain the curvature tensor associated to the metric.

It is evident, geometrically, that there is a great deal of non-uniqueness in the Einstein equations.

If we coordinatize the manifold the equations can be viewed as 10 very complicated PDEs for the 10 component functions of the metric.

The coordinate-free viewpoint: geometry

V an finite dimensional vector space; V^* its dual;

N.B.: there is a canonical identification $V \cong V^{**}$, but not $V \cong V^*$

tensor product $V \otimes W$;

$V \otimes W$ can be thought of as bilinear maps on $V^* \times W^*$ or linear maps from V^* to W or linear maps from W^* to V

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l}$$

$$\cong \text{ multlinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R}$$

Since $V^* \otimes V$ is linear maps from $V \to V$,

$\exists\ \mathrm{tr} : V^* \otimes V \to \mathbb{R}$ given by $\mathrm{tr}(f \otimes v) = f(v)$

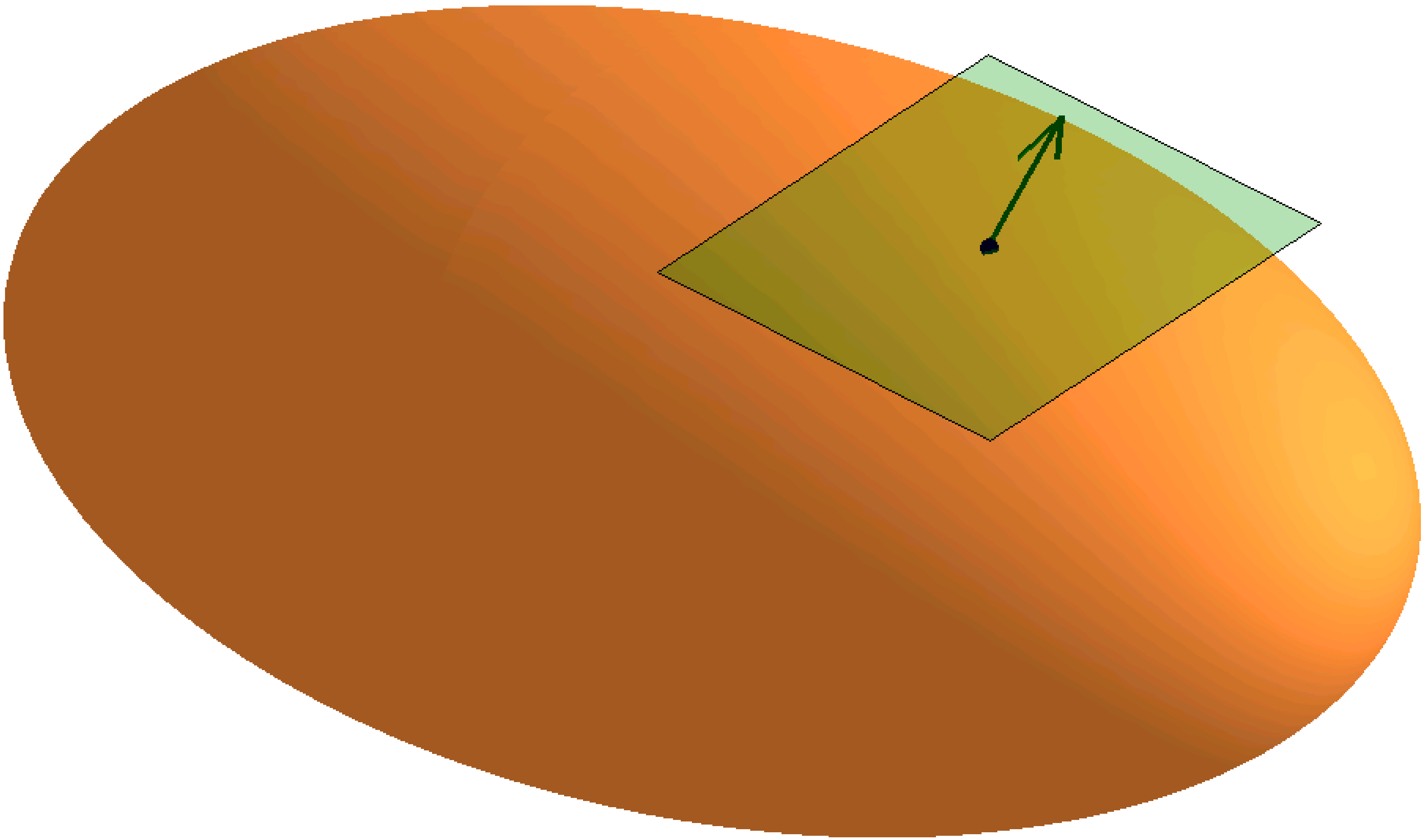
A (pseudo) inner product is a symmetric bilinear map $a : V \times V \to \mathbb{R}$ (so an element of $V^* \otimes V^*$) which is non-degenerate: $a(v, \cdot\,) \neq 0$ if $v \neq 0$.

Given an inner product we can assign every vector a squared length $a(v, v)$. It is not 0 if $v \neq 0$, but it can be negative.

Orthonormal basis: $a(e_i, e_j) = \pm\delta_{ij}$. The number of pluses and minuses is basis-independent, the signature of the inner product.

An inner product establishes a canonical identification $V \cong V^*$

Manifold concepts



M an n-manifold, $p \in M$, T_pM the tangent space of M at p, $(T_pM)^*$ the cotangent space

$$T_p^{(k,l)}M := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_pM^* \otimes \cdots \otimes T_pM^*}_{l}$$

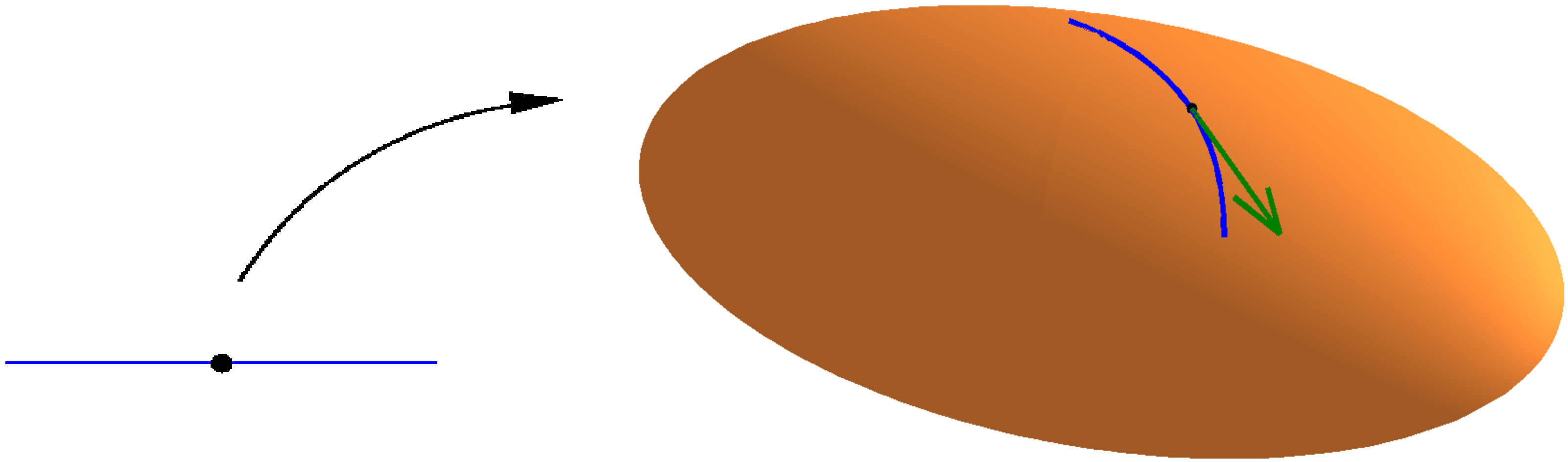
Maps $p \in M \mapsto v_p \in T_p^{(k,l)}M$, are called (k,l)-tensors
$(0,0)$-tensors: functions $M \to \mathbb{R}$; $(1,0)$-tensors: vector fields on M; $(0,1)$-tensors: covector fields on M

A (k,l)-tensor is a machine that at each p takes k tangent covectors and l tangent vectors and returns a number
(multilinear in the (co)vectors, smooth in p).

All physical quantities in relativity are modeled as tensors.

If $\phi : M \to N$ is smooth and $p \in M$, then $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear map. For $v \in T_pM$, $d\phi_p v \in T_{\phi(p)}N$ is also denoted $\phi_* v$, the push-forward of v.

For I an interval about 0, $\gamma : I \to M$ a curve, then $\gamma'(0) := d\gamma_0 1$ is a tangent vector at $\gamma(0)$.

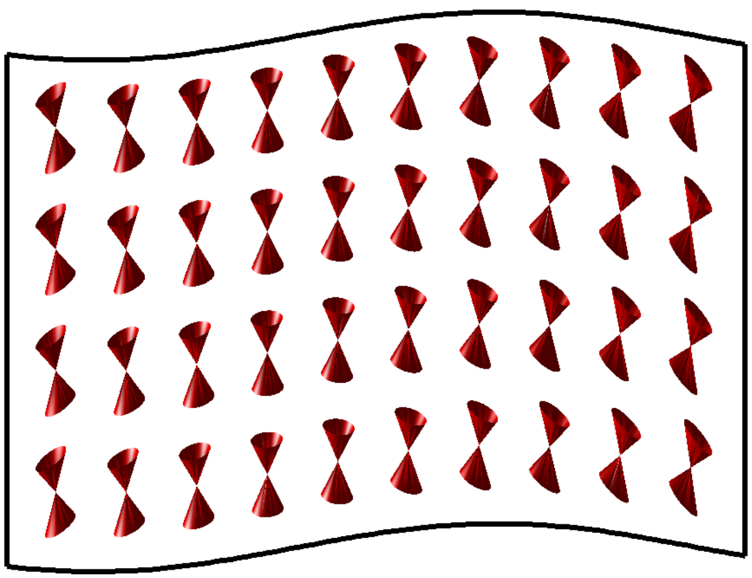


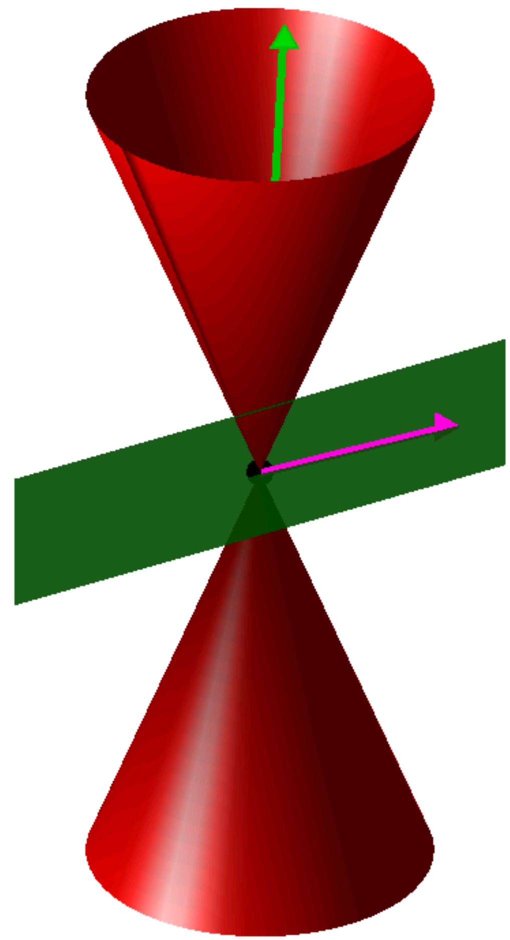
If $f : M \to \mathbb{R}$, then df_p is a linear map T_pM to $\mathbb{R}$, i.e., df is a covector field.

Metrics on manifolds

A pseudo Riemannian metric is a symmetric, non-degenerate $(0,2)$-tensor, i.e., at each point p, an inner product on T_pM

The Einstein equations are concerned with assigning to a manifold a metric with signature $-$ · properties.





For (k,l)-tensors, use symbols adorned with k superscripts and l subscripts a, b, . . .

v^a is a vector field, w_b is a covectorfield, R^d_{abc} is a $(1,3)$-field, etc.

The indices themselves have no meaning (like the $\vec{}$ in $\vec{v}$).

The tensor product of v^a_b and w^{ab}_c is written $v^a_b w^{cd}_e$. Counting sub- and superscripts shows it to be a $(3,2)$-tensor.

The trace of a $(1,1)$-tensor is indicated by a repeated index: v^a_a (Repeated sub-/superscripts aren't counted.)

v^{abc}_{ad} trace of a $(3,2)$-tensor wrt the first covector and vector variables, a $(2,1)$-tensor.

$v_{(ab)} := \frac{1}{2}(v_{ab} + v_{ba})$, the symmetric part of v_{ab}

$v_{[ab]} := \frac{1}{2}(v_{ab} - v_{ba})$, the antisymmetric part of v_{ab}

$v_{(ab)c} := \frac{1}{2}(v_{abc} + v_{bac})$

$v_{(abc)} := \frac{1}{6}(v_{abc} + v_{bca} + v_{cab} + v_{bac} + v_{cba} + v_{acb})$

If a metric g_{ab} is specified, we can identify a covector with a vector. We write v_a for the vector identified with v^b:
$v_a = g_{ab}v^b$

This can apply to one index of many: $g_{ce}w_{ab}^{ed} = w_{abc}^{d}$, or several: $g_{ce}g_{df}w_{ab}^{ef} = w_{abcd}$

Applied to the metric we find g_a^b is the identity δ_a^b, and g^{ab} is the "inverse metric," which can be used to raise indices:
$v^a = g^{ab}v_b$

Given a function $f : M \to \mathbb{R}$ and a vector $V^a \in T_pM$ there is a natural way to define the directional derivative $V^a\nabla_a f$:

$$V^a\nabla_a f(p) = \lim_{\epsilon\to 0} \frac{f(\text{“}p+\epsilon V^a\text{”}) - f(p)}{\epsilon}.$$

By “$p+\epsilon V^a$” we mean $\gamma(\epsilon)$ where $\gamma : \mathbb{R} \to M$ is a curve with $\gamma(0) = p$, $\gamma'(0) = V^a$.

Thus $\nabla_a f$ is a covector field, which we previously called df.

It is not possible to define the directional derivative of a vector field v^b in the same way, because $v^b(\text{“}p+\epsilon V^a\text{”}) - v^b(p)$ involves the difference of vectors in different spaces.

Covariant differentiation and parallel transport

If a metric g_{ab} is specified, this determines a way to parallel transport a vector along a curve. Using this we can define $\nabla_a f^b$. Using the Leibnitz rule this easily extends to tensors of arbitrary variance. In this way we get a linear operator ∇ from (k, l)-tensors to $(k, l+1)$-tensors for all k, l. It satisfies the Leibniz rule, commutes with traces, gives the right result on scalar field, satisfies the symmetry

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f, \qquad f : M \to \mathbb{R}$$

and *is compatible with the metric*:

$$\nabla_a g_{bc} = 0.$$

This characterizes the covariant differentiation operator.

It is not true that the second covariant derivative is symmetric when applied to vectors. Instead

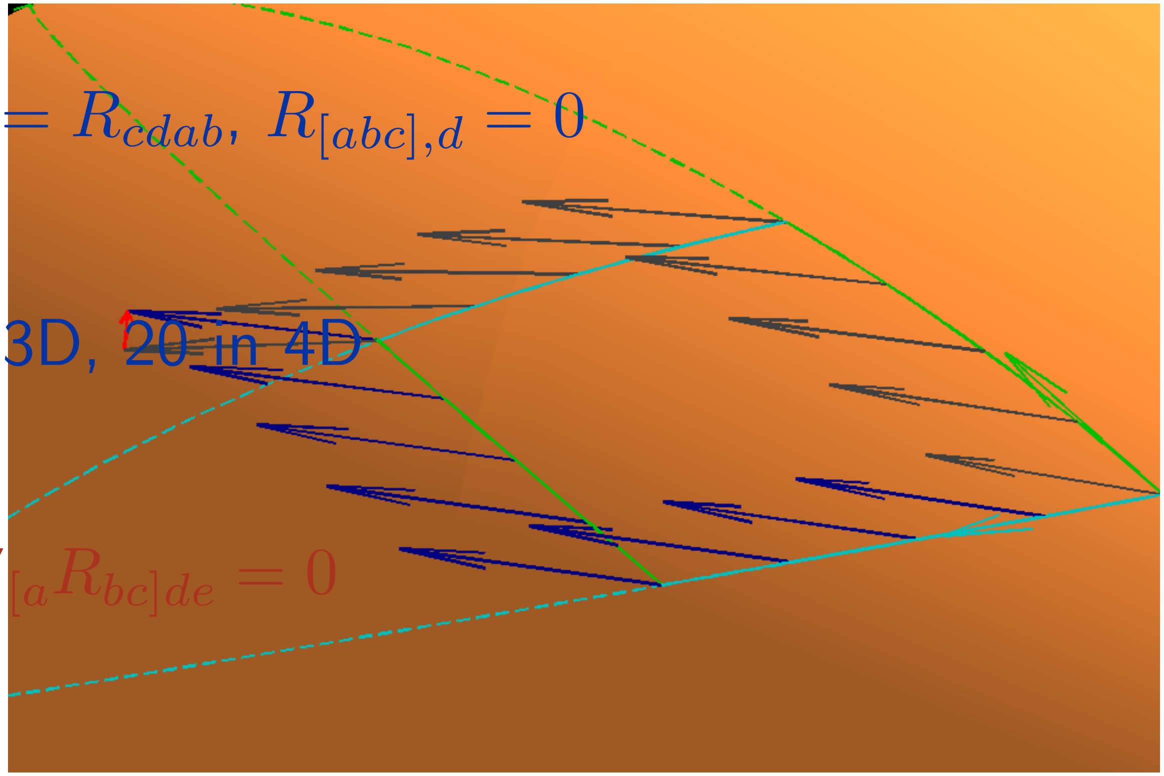
$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^d = \frac{1}{2} R^d_{abc} v^c$$

for some tensor R^d_{abc}, called the Riemann curvature tensor.

$R_{(ab)cd} = 0$, $R_{abcd} = R_{cdab}$, $R_{[abc],d} = 0$

1 DOF in 2D, 6 in 3D, 20 in 4D

Bianchi identity: $\nabla_{[a} R_{bc]de} = 0$



Ricci tensor, scalar curvature, Einstein tensor

The Ricci tensor is the trace of the Riemann tensor:
$R_{ab} = R^{d}_{adb}$

The scalar curvature is its trace: $R = R^a_a = g^{ab}R_{ab}$

The Einstein tensor is $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$.

In 4D G_{ab} has the same trace-free part but opposite trace as R_{ab}: Einstein is trace-reversed Ricci.

By the Bianchi identity, $\nabla^a G_{ab} := g^{ac}\nabla_c G_{ab} = 0$

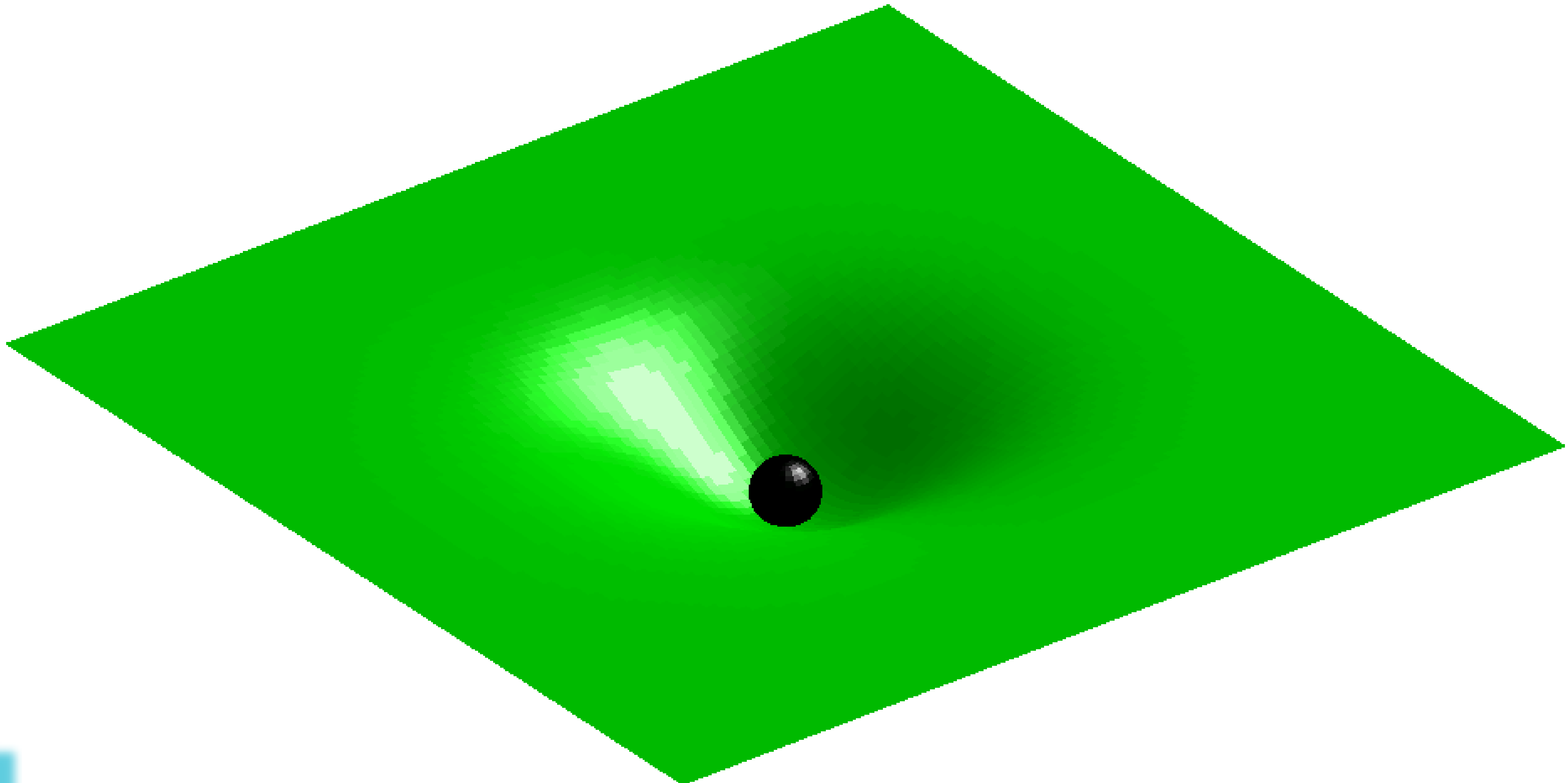
In a vacuum, the Einstein equations are simply

$$G_{ab} = 0$$

or $R_{ab} = 0$.

In GR we are interested in spacetimes, i.e., 4-manifolds endowed with a metric of signature $-+++$ which satisfy the Einstein equations.

If matter is present, then $G_{ab} = kT_{ab}$ where the stress-energy tensor T_{ab} comes from a matter model, $k =$ const. $= 8\pi \mathrm{G}/c^4 = 2 \times 10^{-48}$ sec^2/g cm



If $\phi : M \to N$ is any diffeomorphism of manifolds and we have a metric g on M, then we can push forward to get a metric $\phi_* g$ on N. With this choice of metric ϕ is an isometry. It is obvious that the Riemann/Ricci/scalar/Einstein curvature tensors associated with $\phi_* g$ on N are just the push-forwards of the those associated with g on M. So if g satisfies the vacuum Einstein equations, so does $\phi_* g$.

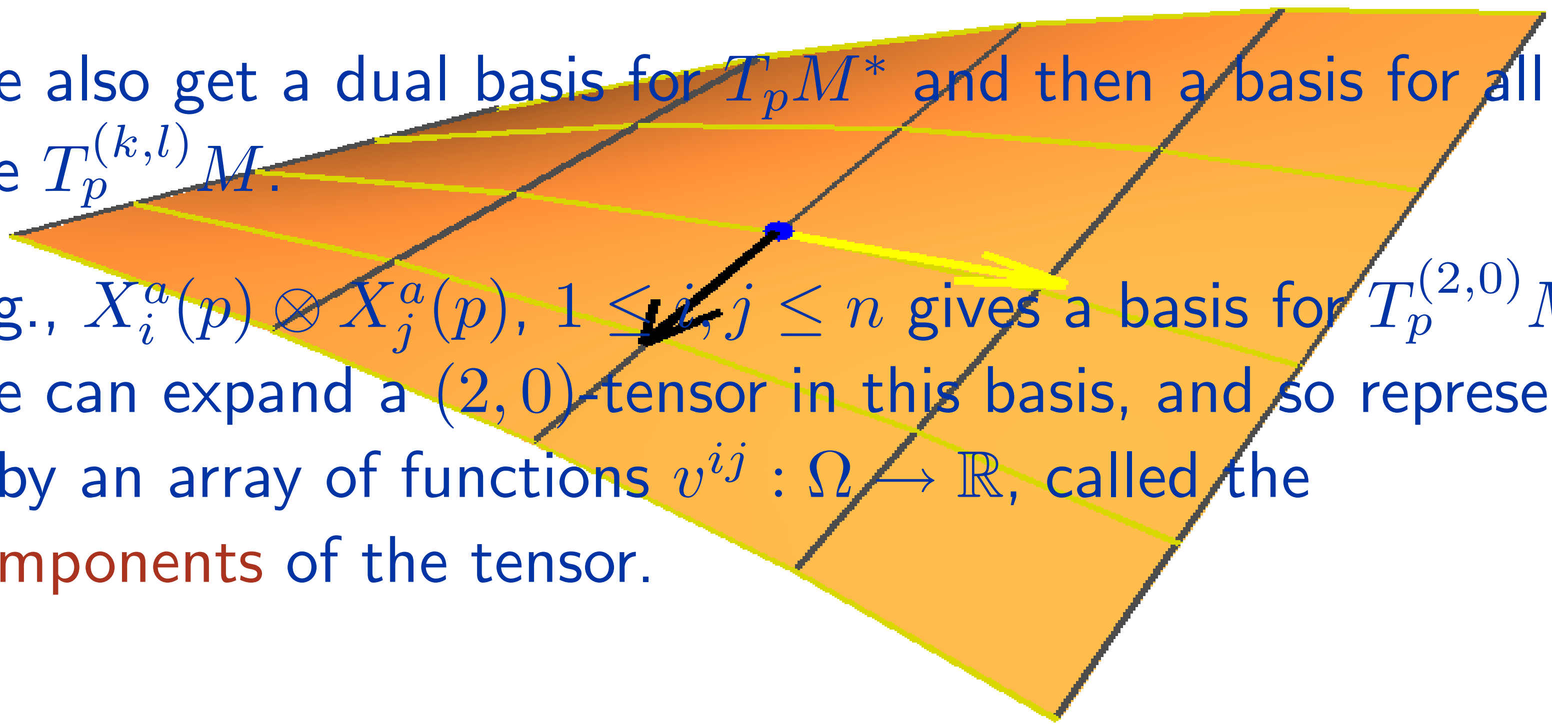
In particular we can map a manifold to itself diffeomorphically, leaving it unchanged in all but a small region. This shows that the Einstein equations plus boundary conditions can never determine a unique metric on a manifold.

Uniqueness can never be for more than an equivalence class of metrics under diffeomorphism.

The coordinate viewpoint:

PDEs

Let $(x^1, \ldots, x^n) : M \to \mathbb{R}^n$ be a diffeomorphism of M (often only part of M) onto $\Omega \subset \mathbb{R}^n$. At each point we can pull back the standard basis of $\mathbb{R}^n$ to a basis for T_pM. This coordinate-dependent choice of basis $(X_1^a(p), \ldots, X_n^a(p))$ at each point is the coordinate frame.



We also get a dual basis for T_pM^* and then a basis for all the $T_p^{(k,l)}M$.

E.g., $X_i^a(p) \otimes X_j^a(p)$, $1 \leq i, j \leq n$ gives a basis for $T_p^{(2,0)}M$. We can expand a $(2,0)$-tensor in this basis, and so represent it by an array of functions $v^{ij} : \Omega \to \mathbb{R}$, called the components of the tensor.

Covariant differentiation in coordinates

If g_{ij} are the components of the metric and v^i are the components of some vector field v^b, then the components of the covariant derivative $\nabla_a v^b$ are

$$\nabla_i v^j = \frac{\partial v^j}{\partial x^i} + \Gamma^j_{ik} v^k,$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

are the Christoffel symbols of the metric in the particular coordinate system. Similar formulas exist for the covariant derivative of tensors of any variance.

Einstein equations in coordinates

$$(g^{ij}) = (g_{ij})^{-1}, \qquad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^m_{jk}\Gamma^l_{mi} - \Gamma^m_{ik}\Gamma^l_{mj}$$

$$R_{ij} = R^l_{ilj}, \qquad R = g^{ij} R_{ij}, \qquad G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

$$G_{ij} = k T_{ij}$$

10 quasilinear second order equations in 10 unknowns and 4 independent variables, 1000s of terms

Gauge freedom in coordinates

Given a second coordinate system $(x'^1, \dots, x'^n) : M \to \Omega'$ we get a second set of component functions g'_{ij} for the same metric.

$$g_{ij}(x) = \frac{\partial \psi^k}{\partial x^i}(x) \frac{\partial \psi^l}{\partial x^j}(x) g'_{kl}(x'),$$

where ψ is $\Omega \to M \to \Omega'$.

(g'_{ij}) satisfies the vacuum Einstein equations iff (g_{ij}) does.

This suggests that roughly 4 of the 10 components g_{ij} can be specified independently of the Einstein equations.

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It is evident, geometrically, that there is a great deal of non-uniqueness in the Einstein equations.

If we coordinatize the manifold the equations can be viewed as 10 very complicated PDEs for the 10 component functions of the metric.

For computational (and other) purposes it is better to view the Einstein equations not as equations for a 4-metric but as equations for a 3-metric that evolves in time. Stay tuned. . .