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A LINEAR SHELL THEORY

BASED ON VARIATIONAL PRINCIPLES

A Thesis in

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by

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Abstract

Under the guidance of variational principles, we derive a two-dimensional shell model, which is a close variant of the classical Naghdi model. From the model solution, approximate stress and displacement fields can be explicitly reconstructed. Convergence of the approximate fields toward the more accurate three-dimensional elasticity solutions is proved. Convergence rates are established. Potential superiority of the Naghdi-type model over the Koiter model is addressed. The condition under which the model might fail is also discussed.

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Chapter 1

Introduction

1.1 Background and motivations

A shell is a three-dimensional elastic body occupying a thin neighborhood of a two-dimensional manifold, which resists deformation owing to the material of which it is made, its shape, and boundary conditions. It is extremely important in structural mechanics and engineering because a well-designed shell can sustain large loads with remarkably little material. For example, before collapsing, a totally clamped spherical shell of thickness $2\,\epsilon$ can hold a strain energy of $O(\epsilon^{-1/3})$ times that which can be tolerated by a flat plate of the same thickness (see page 202). For this reason, shells are a favored structural element in both natural and man-made constructions. While elastic shells can exhibit great strength, their behaviors can also be very difficult to predict, and they can fail in a catastrophic fashion.

Although the deformation of a shell arising in response to given loads and boundary conditions can be accurately captured by solving the three-dimensional elasticity equations, shell theory attempts to provide a two-dimensional representation of the intrinsically three-dimensional phenomenon [34]. There are two reasons to derive a lower dimensional model. One is its simpler mathematical structure. For example, the existence, regularity, bifurcation, and global analysis are by now on firm mathematical grounds for non-linear elastic rods [18]. In contrast, the mathematical theory for non-linear three-dimensional elasticity is much less developed. Another motivation is for numerical simulation. An accurate, fully three-dimensional, simulation of a very thin body is beyond the power of even the most powerful computers and computational techniques. Furthermore, the standard methods of numerical approximation of three-dimensional elastic bodies fail for bodies which are thin in some direction, unless the behavior is resolved in that direction. Thus the need for two-dimensional shell models [5].

Beginning in the late nineteenth century, and especially during the past few decades, there have been intense efforts to derive an accurate dimensionally reduced mathematical theory of shells. Despite much progress, the development of a satisfactory mathematical theory of elastic shells is far from complete. The methodologies for deriving shell models from three-dimensional continuum theories are still being developed, and the relation between different approaches, are not clear. Controversial issues abound. The extremely important question of deriving rigorous mathematical theory relating shell models to more exact three-dimensional models is wide open. A thorough analysis of the mathematical models derived and a rigorous definition of their ranges of applicability is mostly lacking.

There is a huge literature devoted to dimensional reduction in elasticity theory. Several classical approaches are employed in investigations. One approach starts with *a priori* assumptions on the displacement and stress fields based on mechanical considerations, such as the Kirchhoff–Love assumption on the displacements and the kinetic assumption on the stress fields that assumes both the transverse shear and normal stresses are negligible. This approach leads to the biharmonic plate bending model, Koiter shell model, flexural shell model, and many others. Models derived in this way have proved successful in practice, but this approach does not seem to lend itself naturally to an error analysis [2].

Another approach is through a formal asymptotic analysis in which the thickness of the elastic body is viewed as a small parameter. By expanding the three-dimensional elasticity equation with respect to the thickness, the leading terms in the expansion are used to define lower dimensional models. This approach leads to limiting models describing the zero thickness limit situation, among which are the limiting flexural and membrane models, depending on ad hoc assumptions on the applied forces, the shell geometry, and boundary conditions. These asymptotic methods only lead to the limiting models. It does not seem to be possible to derive the better Koiter and Naghdi models by this approach. (Taking more terms in the asymptotic expansion does not lead to a dimensionally reduced model.) See [18] for a comprehensive treatment of this approach.

A third approach is by variational methods. Solution of the three-dimensional equation can be characterized by variational principles or weak formulations. An approximation is determined by restricting to a trial space of functions that are finite dimensional with respect to the transverse variable. By its very nature, this approach leads to models that yield a displacement field or a stress field determined by finitely many functions of two variables. Thus the dimension is reduced. In this approach, the two energies principle, or the Prager–Synge theorem [54], plays a fundamental role in the model validation. To apply the two energies principle, we must have a statically admissible stress field and a kinematically admissible displacement field. The latter is usually easy to come by, but the former might be formidable to obtain. The two energies principle is particularly suited to analyzing complementary energy variational models, which automatically yield statically admissible stress fields.

The application of the two energies principle to justify plate theory was initiated in the pioneering work of Morgenstern [47], where it was used to prove the convergence of the biharmonic model of plate bending when the thickness tends to zero. The statically admissible stress field and the kinematically admissible displacement field were constructed based on the biharmonic solution in an ad hoc fashion, as needed for the convergence proof. Following this work, substantial efforts have been made to modify the justification of the classical plate bending models, see [48], [51], and [57]. In the same spirit, Gol'denveizer [29], Sensenig [56], Koiter [33], Mathúna [46], and many others considered the error estimates for shell theories. In these latter works, the stress fields constructed from the model solutions were only approximately admissible, and the justifications obtained were largely formal.

Due to the formidable difficulty involved in the construction of an admissible stress field based on the solution of a known model, it seems a better choice to reconsider the derivation of the model while keeping in mind the construction of the statically admissible stress field as a primary goal. Based on the Hellinger–Reissner variational formulations of the three-dimensional elasticity, a systematic procedure of dimensional reduction for plate problems was developed in [2]. In this approach both the stress and displacement fields were restricted to subspaces in which functions depend on the transverse coordinate polynomially. The derivation based on the second Hellinger–Reissner principle not only led to the well known Reissner–Mindlin plate model but also furnished an admissible stress field and so naturally led to a rigorous justification of the model by the two energies principle. This approach is not easily extensible to shell problems. Due to the curved shape of a shell, if this approach were carried over and the subspaces were chosen to be composed of functions depending on the transverse coordinate polynomially, the polynomials would be of conspicuously higher order. The resulting model would contain so many unknowns that it would be nearly as untractable as the three-dimensional model.

In this work we derive and rigorously justify a two-dimensional shell model guided by the variational principles.

1.2 Organization of this thesis

We consider the modeling of the deformation arising in response to applied forces and boundary conditions of an arbitrary thin curved shell, which is made of isotropic and homogeneous elastic material whose Lamé coefficients are λ and μ. The shell is clamped on a part of its lateral face and is loaded by a surface force on the remaining part of the lateral face. The shell is subjected to surface tractions on the upper and lower surfaces and loaded by a body force. We take the three-dimensional linearized elasticity equation as the supermodel and approximate it by a two-dimensional model. The lower dimensional model will be justified by proving convergence and establishing the convergence rate of the model solution to the solution of the three-dimensional elasticity equation in the relative energy norm under some assumptions on the applied forces. Conditions under which the model might fail will be discussed. The two energies principle supplies important guidance for the construction of the model.

Throughout the thesis, Greek subscripts and superscripts, except ϵ, which is reserved for the half-thickness of the shell, always take their values in $\{1, 2\}$, while Latin scripts always belong to the set $\{1, 2, 3\}$. Summation convention with respect to repeated superscripts and subscripts will be used together with these rules. We usually use lower case Latin letters with an undertilde, as $\underset{\sim}{v}$, to denote two-dimensional vectors. Lower case Greek letters with double undertildes denote two-dimensional second order tensors, as $\underset{\approx}{\sigma}$. However, the fundamental forms on the shell middle surface will be denoted by lower case Latin letters. We use boldface Latin letters to denote three-dimensional vectors and boldface Greek letters second order three-dimensional tensors. Vectors and tensors will be given in terms of their covariant components, or contravariant components, or mixed components.

The notation $P \simeq Q$ means there exist constants C_1 and C_2 independent of ϵ, P, and Q such that $C_1 P \le Q \le C_2 P$. The notation $P \lesssim Q$ means there exists a constant C independent of ϵ, P, and Q such that $P \le CQ$.

Chapters 2–6 form the main body of the thesis, with Chapters 2 and 5 treating two special kinds of shells, namely, the plane strain cylindrical shells and spherical shells, respectively; Chapter 6 treating general shells; and Chapters 3 and 4 containing results needed for the analysis. The reason we treat cylindrical and spherical shells separately is that for these special shell problems, we can construct statically admissible stress fields and kinematically admissible displacement fields, so that we can use the two energies principle to justify the models by bounding the constitutive residuals. As a consequence, stronger convergence results can be obtained for these cases. These two special shells provide examples for all kinds of shells as classified in Section 3.5. For the general shells treated in Chapter 6, precisely admissible stress fields are no longer possible to construct. The derivation yields an almost admissible stress field with small residuals in the equilibrium equation and lateral traction boundary condition. The two energies principle can not be directly used to justify the model. As an alternative, we establish an integration identity to incorporate all these residuals so that we can bound the model error by estimating these residuals.

All the models we derive can be written in variational forms, in which the flexural energy, membrane energy, and shear energy are combined together in the total strain energy. Contributions of the component energies are weighted by factors that depend on ϵ. Chapter 3 is devoted to the mathematical analysis of such ϵ-dependent problems on an abstract level. In this chapter, we classify the model and analyze the asymptotic behavior of the model solution when the shell thickness approaches zero. The range of applicability of the derived model will also be discussed on the abstract level. The rigorous validation of the shell model crucially hinges on these analyses.

In Chapter 4, we briefly summarize the three-dimensional linearized elasticity theory expressed in the curvilinear coordinates on a thin shell. We also derive some formulas that can substantially simplify calculations. Finally, in Chapter 7, we will discuss the relations between our theory and other existing shell theories. In the remainder of this introduction, we will describe the principal results of the following chapters.

1.3 Principal results

In this section we summerize the key results of Chapters 2, 5, and 6.

1.3.1 Plane strain cylindrical shells

In Chapter 2 we consider the simplest case of plane strain cylindrical shells. In this case, the three-dimensional problem is essentially a two-dimensional problem defined on a cross-section, so the dimensionally reduced model should be one-dimensional. We assume that the cylindrical shell is clamped on the two lateral sides, subjected to surface forces on the upper and lower surfaces, and loaded by a body force.

Let the middle curve of a cross-section of the cylindrical shell be parameterized by its arc length variable $x \in [0, L]$. Our model can be written as a one-dimensional variational problem defined on the space $H = [H_0^1(0, L)]^3$. The solution of the model is composed of three single variable functions that approximately describe the shell deformation arising in response to the applied forces and boundary conditions. We introduce the following operators. For any $(\theta, u, w) \in H$, we define

$$\gamma(u, w) = \partial u - bw, \quad \rho(\theta, u, w) = \partial\theta + b(\partial u - bw), \quad \tau(\theta, u, w) = \theta + \partial w + bu,$$

which give the membrane strain, flexural strain, and transverse shear strain engendered by the displacement functions (θ, u, w). Here b is the curvature of the middle curve, which is a function of the arc length parameter, and $\partial = d/dx$.

The model (cf., (2.3.2) below) reads: Find $(\theta^\epsilon, u^\epsilon, w^\epsilon) \in H$, such that

$$\frac{1}{3}\epsilon^2(2\mu + \lambda^\star)\int_0^L \rho(\theta^\epsilon, u^\epsilon, w^\epsilon)\rho(\phi, y, z)dx$$

$$+ (2\mu + \lambda^\star) \int_0^L \gamma(u^\epsilon, w^\epsilon)\gamma(y,z)dx + \frac{5}{6}\mu \int_0^L \tau(\theta^\epsilon, u^\epsilon, w^\epsilon)\tau(\phi, y, z)dx$$

$$= \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\phi, y, z)\rangle, \ \forall(\phi, y, z) \in H,$$

in which

$$\lambda^\star = \frac{2\mu\lambda}{2\mu + \lambda}$$

and the loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$ is explicitly expressible in terms of the applied force functions, cf., (2.3.3), (2.3.4). We show that the solution of this one-dimensional model uniquely exists. The three single variable functions θ^ϵ, u^ϵ, and w^ϵ that comprise the model solution describe the rotations of straight fibers normal to the middle curve, the tangential displacements, and transverse displacements of points on the middle curve, respectively.

In addition to the model, in Section 2.4 we give formulae to reconstruct a tensor field $\underset{\approx}{\sigma}$ and a vector field $\underset{\sim}{v}$ from the model solution on the shell cross-section, see equations (2.4.1), (2.4.3), (2.4.7), and (2.4.8). The model and reconstruction formulae are designed to have the following properties:

(1) $\underset{\approx}{\sigma}$ is a statically admissible stress field (see Section 2.4.1).

(2) $\underset{\sim}{v}$ is a kinematically admissible displacement field (see Section 2.4.2).

(3) The terms of leading order in ϵ in the constitutive residual $A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma} - \chi_{\alpha\beta}(\underset{\sim}{v})$ vanish, so the constitutive residual may be shown to be small as $\epsilon \to 0$ (see Section 2.4.3).

This allows a bound on the errors of $\underset{\approx}{\sigma}$ and $\underset{\sim}{v}$ by the two energies principle. Under the loading assumptions (2.3.6) and (2.5.1), we prove the inequality

$$\frac{\|\underset{\approx}{\sigma}^* - \underset{\approx}{\sigma}\|_{E^\epsilon} + \|\underset{\approx}{\chi}(\underset{\sim}{v}^*) - \underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}}{\|\underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}} \lesssim \epsilon^{1/2},$$

in which $\underset{\approx}{\sigma}^*$ is the stress field and $\underset{\sim}{v}^*$ the displacement field arising in the shell determined from the two-dimensional elasticity equations. The norm $\|\cdot\|_{E^\epsilon}$ is the energy norm of the strain or stress field.

1.3.2 Spherical shells

For spherical shells, we derive the model by a similar method. We assume the middle surface of the shell is a portion of a sphere of radius R. The shell is clamped on a part of its lateral face, and subjected to surface force on the remaining part of the lateral face whose density is linearly dependent on the transverse variable. The shell is subjected to surface forces on the upper and lower surfaces, and loaded by a body force whose density is assumed to be constant in the transverse coordinate. The middle surface is parameterized by a mapping from a domain $\omega \subset \mathbb{R}^2$ onto it. The boundary $\partial\omega$ is divided as $\partial\omega = \partial_D\omega \cup \partial_T\omega$ giving the clamping and traction parts of the the lateral face of the shell. The model is a two-dimensional variational problem defined on the space $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$. The solution of the model is composed of five two variable functions that can approximately describe the shell displacement arising in response to the applied loads and boundary conditions. For $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$, we define

$$\gamma_{\alpha\beta}(\underset{\sim}{u}, w) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b a_{\alpha\beta} w,$$

$$\rho_{\alpha\beta}(\underset{\sim}{\theta}) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}), \quad \tau_{\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \theta_{\beta} + \partial_{\beta} w + b u_{\beta},$$

which give the membrane, flexural, and transverse shear strains engendered by the displacement functions $(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$. Here, $a_{\alpha\beta}$ is the covariant metric tensor and $b = -1/R$ is the curvature of the middle surface. The model (cf., (5.3.2)) reads: Find $(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \in H$, such that

$$\begin{aligned} &\frac{1}{3}\epsilon^2 \int_{\omega} a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}(\underset{\sim}{\theta}^{\epsilon}) \rho_{\alpha\beta}(\underset{\sim}{\phi}) \sqrt{a} d\underset{\sim}{x} \\ &+ \int_{\omega} a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \gamma_{\alpha\beta}(\underset{\sim}{v}, z) \sqrt{a} d\underset{\sim}{x} + \frac{5}{6}\mu \int_{\omega} a^{\alpha\beta} \tau_{\beta}(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \tau_{\alpha}(\underset{\sim}{\phi}, \underset{\sim}{v}, z) \sqrt{a} d\underset{\sim}{x} \\ &\qquad = \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \rangle, \ \forall \ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H \end{aligned}$$

where $a^{\alpha\beta}$ is the contravariant metric tensor of the middle surface and

$$a^{\alpha\beta\lambda\gamma} = 2\mu a^{\alpha\lambda} a^{\beta\gamma} + \lambda^{\star} a^{\alpha\beta} a^{\lambda\gamma} \tag{1.3.1}$$

is the two-dimensional elasticity tensor of the shell. The resultant loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$ can be explicitly expressed in terms of the applied force functions, cf., (5.3.3), (5.3.4). This model has a unique solution if the resultant loading functional is in the dual space of H. This condition is satisfied if the applied force functions satisfy the condition (5.3.6). The unique solution $(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon})$ describes the normal straight fiber rotations, middle surface tangential displacement and transverse displacement, respectively. A statically admissible stress field and a kinematically admissible displacement field can be reconstructed from the model solution. We prove the convergence and establish the convergence rate of the model solution to the three-dimensional solution by estimating the constitutive residual.

1.3.3 General shells

For a general shell, except for some smoothness requirements, we do not impose any restriction on the geometry of the shell middle surface or the shape of its lateral boundary. The shell is assumed to be clamped on a part of its lateral surface and loaded by a surface force on the remaining part. The shell is subjected to surface forces on the upper and lower surfaces, and loaded by a body force.

The model is constructed in the vein of the model constructions for the special shells in the Chapters 2 and 5. The main difficulty to overcome is that our model derivation does not yield a statically admissible stress field. Therefore, the two energies principle can not be directly used to justify the model. Even so, we can reconstruct a stress field that is almost admissible with small residuals in the equilibrium equation and lateral traction boundary condition. And we will establish an integration identity (6.3.17) to incorporate the equilibrium residual and the lateral traction boundary condition residual. This identity plays the role of the two energies principle in the general shell theory.

Let the middle surface of the shell be parameterized by a mapping from the domain $\omega \subset \mathbb{R}^2$ onto it. Corresponding to the clamping and traction parts of the lateral face, the boundary of ω is divided as $\partial\omega = \partial_D\omega \cup \partial_T\omega$. In this curvilinear coordinates, the fundamental forms on the shell middle surface are denoted by $a_{\alpha\beta}$, $b_{\alpha\beta}$, and $c_{\alpha\beta}$. The mixed curvature tensor is denoted by b^α_β. The model is a two-dimensional variational problem defined on the space $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$. The solution of the model is composed of five two variable functions that can approximately describe the shell displacement arising in response to the applied loads and boundary conditions. For $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$, we define the following two-dimensional tensors.

$$\gamma_{\alpha\beta}(\underset{\sim}{u}, w) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w,$$

$$\rho_{\alpha\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + \frac{1}{2}(b^\lambda_\beta u_{\alpha|\lambda} + b^\lambda_\alpha u_{\beta|\lambda}) - c_{\alpha\beta}w,$$

$$\tau_\beta(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = b^\lambda_\beta u_\lambda + \theta_\beta + \partial_\beta w.$$

These two-dimensional tensor- and vector-valued functions give the membrane strain, flexural strain, and transverse shear strain engendered by the displacement functions $(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$, respectively. The model (cf., (6.2.4)) reads: Find $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \in H$, such that

$$\begin{aligned}
&\frac{1}{3}\epsilon^2 \int_\omega a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \rho_{\alpha\beta}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x} \\
&+ \int_\omega a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\underset{\sim}{u}^\epsilon, w^\epsilon) \gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x} + \frac{5}{6}\mu \int_\omega a^{\alpha\beta} \tau_\beta(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x} \\
&\qquad = \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle, \ \forall (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H,
\end{aligned}$$

in which the fourth order two-dimensional contravariant tensor $a^{\alpha\beta\lambda\gamma}$ is the elastic tensor of the shell, defined by the formula (1.3.1). The resultant loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$

can be explicitly expressed in terms of the applied force functions, cf., (6.2.5), (6.2.6). This model has a unique solution if the resultant loading functional is in the dual space of H, a condition that can be easily satisfied. From the model solution, we can reconstruct a stress field $\boldsymbol{\sigma}$ by explicitly giving its contravariant components. By a correction to the transverse deflection, we can define a displacement field $\boldsymbol{v}$ by giving its covariant components. Under some conditions, we will prove the convergence of both $\boldsymbol{\sigma}$ and $\boldsymbol{v}$ to the stress and displacement fields determined from the three-dimensional elasticity equation by using the aforementioned identity and bounding the constitutive residual, equilibrium equation residual and lateral traction boundary condition residual.

The model is a close variant of the classical Naghdi shell model. This model differs from the generally accepted Naghdi model in three ways. First, the resultant loading functional has a somewhat more involved form. Second, the coefficient of the shear term is 5/6 rather than the usual value 1. The "best" choice for this coefficient seems an unresolved issue for shells. When the shell is flat, the model degenerates to the Reissner–Mindlin plate bending and stretching models for which the corresponding value 5/6 is often accepted as the best, see [55] and [2]. The third, and most significant, difference is in the expression of the flexural strain $\rho_{\alpha\beta}$. The relation between our definition and that of Naghdi's ($\rho^N_{\alpha\beta}$) is

$$\rho_{\alpha\beta} = \rho^N_{\alpha\beta} + b^\lambda_\alpha \gamma_{\lambda\beta} + b^\gamma_\beta \gamma_{\gamma\alpha}.$$

We will see that the change of the flexural strain expression appears to be necessary to make the constitutive residual small in some cases (see Remark 6.3.2). In most cases, this difference does not affect the convergence of the model solution to the three-dimensional solution.

When the general shell model is applied to spherical shells, we obtain a spherical shell model slightly different from what we derived in Chapter 5 both in the form of the flexural strain and in the resultant loading functional. The convergence properties of these two spherical shell models are the same. What we can learn from this discrepancy is that the model can be changed, but the resultant loading functional must be changed accordingly, otherwise a variation in the form of a model might lead to divergence.

To prove convergence, we need to make an assumption on the dependence of the applied force functions on the shell thickness. We will assume that all the applied force functions that are explicitly involved in the resultant loading functional are independent of ϵ. Under this assumption, by properly defining function spaces and operators, the shell models can be abstracted to the variational problem:

$$\begin{gathered}\epsilon^2(Au,Av)_U+(Bu,Bv)_V=\langle f_0+\epsilon^2 f_1,v\rangle_{H^*\times H},\\ u\in H,\quad \forall\, v\in H,\end{gathered} \tag{1.3.2}$$

where H, U, and V are Hilbert spaces. The functionals f_0 and f_1 are independent of ϵ. The linear bounded operators A and B are from H to U and V respectively, with the property

$$\|Au\|_U^2+\|Bu\|_V^2\simeq\|u\|_H^2\quad\forall\, u\in H.$$

We can assume that the range W of the operator B is dense in V, and equip W with a norm to make it a Hilbert space. For the plane strain cylindrical shell model, we can prove that the operator B has closed range. This special property substantially simplifies the analysis of behavior of the model solution and significantly strengthes the convergence results.

The asymptotic behavior of the solution of this abstract problem is mostly determined by the leading term f_0 in the loading functional. We classify the problem as a flexural shell problem if $f_0|_{\ker B} \neq 0$. For flexural shells, after scaling the applied forces, the model can be viewed as the penalization of the limiting flexural shell model, which is constrained on $\ker B$ and independent of ϵ. The behavior of the model solution and its convergence property to the three-dimensional solution crucially hinge on the regularity of the Lagrange multiplier ξ^0 of this constrained limiting problem. Without any extra assumption, we have $\xi^0 \in W^*$ and the convergence

$$\lim_{\epsilon \to 0} \frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0, \tag{1.3.3}$$

in which $\boldsymbol{\sigma}^*$ is the stress field and $\boldsymbol{v}^*$ is the displacement field determined from the three-dimensional elasticity. The norm is the energy norm and $\boldsymbol{\chi}(\boldsymbol{v})$ is the three-dimensional strain field engendered by the displacement $\boldsymbol{v}$.

The convergence rate essentially depends on the position of ξ^0 between V^* and W^*. Under the assumption (6.5.14), we can prove the inequality

$$\frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^\theta, \tag{1.3.4}$$

in which $\theta \in [0, 1]$. Note that the case of $\theta = 0$ corresponds to the situation that ξ^0 is only in W^*. The previous convergence result can not be deduced from this result on the convergence rate.

If $f_0|_{\ker B} = 0$, by the closed range theorem in functional analysis, there exists a unique $\zeta_*^0 \in W^*$ such that the leading term of the resultant loading functional can be reformulated as

$$\langle f_0, v \rangle_{H^* \times H} = \langle \zeta_*^0, Bv \rangle_{W^* \times W} \quad \forall\, v \in H.$$

If we only have $\zeta_*^0 \in W^*$, we can not prove any convergence. Very likely, the model diverges in the energy norm in this case. If $\zeta_*^0 \in V^*$, the abstract problem will be called a membrane–shear problem. This condition is a necessary requirement for us to prove the convergence of the model solution to the three-dimensional solution. Under this condition and the assumption that the applied forces are admissible (the admissible assumption on the applied forces is not needed for spherical shells), we can prove a convergence of the form (1.3.3). The convergence rate is determined by where ζ^0, the Riesz representation of ζ_*^0 in V, stands between W and V. For a totally clamped elliptic shell, which is a special example of membrane–shear shells, under some smoothness assumption on the shell data in the Sobolev sense, we prove the convergence rate

$$\frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^{1/6}.$$

If the odd part of the tangential surface forces vanishes, the convergence rate $O(\epsilon^{1/5})$ can be proved.

The condition $\zeta_*^0 \in V^*$ is essentially equivalent to the existence condition for a solution of the "generalized membrane" shell model defined in [18]. This condition is trivially satisfied for shear dominated plane strain cylindrical shells. For shear dominated plate bending, the condition is satisfied as long as the loading function belongs to L_2. The condition is acceptable for stiff parabolic shells and stiff hyperbolic shells. It can be satisfied for a totally clamped elliptic shell if the shell data are fairly smooth in the Sobolev sense. But it imposes a stringent restriction for a partially clamped elliptic shell, in which case even if the shell data are infinitely smooth, the condition might not be satisfied. If the condition is not satisfied, although the model solution always exists, a rigorous relation to the three-dimensional solution is completely lacking.

To reveal the potential advantages of using the Naghdi-type model, we need a different assumption on the applied force functions. Specifically, we assume that the odd part of the applied surface forces has a bigger magnitude than what usually assumed. Under this assumption and in the convergent case of membrane–shear shells, the model solution violates the Kirchhoff–Love hypothesis on which the Koiter shell model were based. Therefore it can not converge.

Finally, in the last chapter, we give justifications for other linear shell models based on the convergence theorems proved for the general shell model, and we will show that under the usual loading assumption, the differences between our model and other models are not significant.

For lack of space, we excluded the model derivations. We will directly present the models and address the more important issue of rigorous justifications.

Chapter 2

Plane strain cylindrical shell model

2.1 Introduction

The shell problem of this chapter is a special example of general shells. The mathematical structure of the derived model is much simpler and we can get much stronger results on the model convergence. Although the problem is simple, it reveals our basic strategy to tackle the general problem.

We consider a 3D elastic body that is an infinitely long cylinder whose cross section is a curvilinear thin rectangle. The body is clamped on the two lateral sides and subjected to surface traction forces on the upper and lower surfaces and loaded by a body force. The applied forces are assumed to be in the sectional plane. Under these assumptions, the elasticity problem is a plane strain problem and can be fully described by a 2D problem defined on a cross section. We assume that the width $2\,\epsilon$ of the sectional curvilinear rectangle is much smaller than its length, so the cylinder is a thin shell.

When the shell is thin, it is reasonable to approximately reduce the 2D elasticity problem to a 1D problem defined on the middle curve of a cross section. A system of ordinary differential equations defined on the middle curve that can effectively capture the displacement and stress of the shell arising in response to the applied forces and boundary conditions will be the desired shell model.

The model, which is a close variant of the Naghdi shell model, is constructed under the guidance of the two energies principle. The plane strain elasticity problem and the two energies principle will be briefly described in section 2.2. The model will be presented and the existence and uniqueness of its solution will be proved in Section 2.3. We reconstruct the admissible stress and displacement fields from the model solution and compute the constitutive residual in Section 2.4. In Section 2.5, we analyze the asymptotic behavior of the model solution and prove the convergence theorem.

Our conclusion is that when the limiting flexural model has a nonzero solution, our model solution converges to the exact solution at the rate of $\epsilon^{1/2}$ in the relative energy norm. In this case, the model is just as good as the limiting flexural model and Koiter model. When the solution of the limiting flexural model is zero, our model gives a solution that can capture the membrane and shear deformations, and the convergence rate in the relative energy norm is still $\epsilon^{1/2}$. The non-vanishing transverse shear deformation violates the Kirchhoff–Love hypothesis in this case. Finally, to emphasize the necessity of using the Naghdi-type model in some cases, we give two examples in which the deformations are shear-dominated, which can be very well captured by our model, but is totally missed by the limiting flexural model and the Koiter model.

2.2 Plane strain cylindrical shells

Since the cross section of the cylindrical shell is a curvilinear rectangle, it is advantageous to work with curvilinear coordinates. In this section we briefly describe the plane strain elasticity theory in curvilinear coordinates for a cylindrical shell.

2.2.1 Curvilinear coordinates on a plane domain

Let $\omega \subset \mathbb{R}^2$ be an open domain, and (x_1, x_2) be the Cartesian coordinates of a generic point in it. Let $\mathbf{\Phi} : \bar{\omega} \to \mathbb{R}^2$ be an injective mapping. We assume that $\Omega = \mathbf{\Phi}(\omega)$ is a connected open domain and $\partial\Omega = \mathbf{\Phi}(\partial\omega)$. The pair of numbers (x_1, x_2) then furnish the curvilinear coordinates on Ω. At any point along the coordinate lines, the tangential vectors $\boldsymbol{g}_\alpha = \partial\mathbf{\Phi}/\partial x_\alpha$ form the covariant basis. The covariant components $g_{\alpha\beta}$ of the metric tensor are given by $g_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta$. The contravariant basis vectors are determined by the relation $\boldsymbol{g}^\alpha \cdot \boldsymbol{g}_\beta = \delta^\alpha_\beta$. The contravariant components of the metric tensor are $g^{\alpha\beta} = \boldsymbol{g}^\alpha \cdot \boldsymbol{g}^\beta$. Note that $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$. The Christoffel symbols are defined by $\Gamma^{*\lambda}_{\alpha\beta} = \boldsymbol{g}^\lambda \cdot \partial_\beta \boldsymbol{g}_\alpha$.

Any vector field $\underset{\sim}{v}$ defined on Ω can be expressed in terms of its covariant components v_α or contravariant components v^α by $\underset{\sim}{v} = v_\alpha \boldsymbol{g}^\alpha = v^\alpha \boldsymbol{g}_\alpha$. Any second-order tensor field $\underset{\approx}{\sigma}$ can be expressed in terms of its contravariant $\sigma^{\alpha\beta}$ or covariant components $\sigma_{\alpha\beta}$ by $\underset{\approx}{\sigma} = \sigma^{\alpha\beta} \boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta = \sigma_{\alpha\beta} \boldsymbol{g}^\alpha \otimes \boldsymbol{g}^\beta$.

The covariant derivative, a second order tensor field, of a vector field $\underset{\sim}{v}$ is defined in terms of covariant components by

$$v_{\alpha\|\beta} = \partial_\beta v_\alpha - \Gamma^{*\lambda}_{\alpha\beta} v_\lambda, \tag{2.2.1}$$

which is the gradient of the vector field.

The covariant derivative of a tensor field with contravariant components $\sigma^{\alpha\beta}$ is defined by

$$\sigma^{\alpha\beta}\|_\lambda = \partial_\lambda \sigma^{\alpha\beta} + \Gamma^{*\alpha}_{\lambda\gamma} \sigma^{\gamma\beta} + \Gamma^{*\beta}_{\lambda\gamma} \sigma^{\alpha\gamma}, \tag{2.2.2}$$

which are mixed components of a third order tensor field. The row divergence of the tensor field $\sigma^{\alpha\beta}$ is a vector field resulting from a contraction of this third order tensor,

$$\operatorname{div} \underset{\approx}{\sigma} = \sigma^{\alpha\beta}\|_{\beta} = \partial_{\beta}\sigma^{\alpha\beta} + \Gamma^{*\alpha}_{\beta\gamma}\sigma^{\beta\gamma} + \Gamma^{*\beta}_{\beta\gamma}\sigma^{\alpha\gamma}. \tag{2.2.3}$$

The components of a vector or tensor field defined over Ω can be viewed as functions defined on the coordinate domain ω.

2.2.2 Plane strain elasticity

Let an infinitely long cylindrical elastic body occupying the 3D domain $\Omega \times (-\infty, \infty) \subset \mathbb{R}^3$ be clamped on a part of its surface $\partial_D\Omega \times (-\infty, \infty)$. On the remaining part of the surface $\partial_T\Omega \times (-\infty, \infty)$, the body is subjected to the surface traction force whose density $\underset{\sim}{p}$ is in the Ω-plane and independent of the longitudinal direction. If the applied body force $\underset{\sim}{q}$ is also assumed to be in the Ω-plane and independent of the longitudinal direction, the displacement of the body arising in response to the applied forces and clamping boundary condition will be in the plane of Ω and constant in the longitudinal direction. The displacement can be represented by a 2D vector field $\underset{\sim}{v}$ and the strain by a 2D tensor field $\underset{\approx}{\chi}$ defined on Ω. The stress field can also be treated as a 2D tensor field $\underset{\approx}{\sigma}$ that is composed of the in-plane components. Although the stress component in the direction normal to the Ω-plane does not vanish, it is totally determined by the in-plane stress components.

The following five equations (2.2.4–2.2.8) constitute the theory of plane strain elasticity. The theory includes the geometric equation

$$\chi_{\alpha\beta}(\underset{\sim}{v}) = \frac{1}{2}(v_{\alpha\|\beta} + v_{\beta\|\alpha}) \tag{2.2.4}$$

and the constitutive equation

$$\sigma^{\alpha\beta} = C^{\alpha\beta\lambda\gamma}\chi_{\lambda\gamma}, \quad \text{or} \quad \chi_{\alpha\beta} = A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma}, \tag{2.2.5}$$

where the fourth order tensors $C^{\alpha\beta\lambda\gamma}$ and $A_{\alpha\beta\lambda\gamma}$ are the plane strain elasticity tensor and the compliance tensor respectively, given by

$$C^{\alpha\beta\lambda\gamma} = 2\mu g^{\alpha\lambda}g^{\beta\gamma} + \lambda g^{\alpha\beta}g^{\lambda\gamma}$$

and

$$A_{\alpha\beta\lambda\gamma} = \frac{1}{2\mu}[g_{\alpha\lambda}g_{\beta\gamma} - \frac{\lambda}{2(\mu+\lambda)}g_{\alpha\beta}g_{\lambda\gamma}],$$

in which λ and μ are the Lamé coefficients of the elastic material comprising the cylinder. To describe the equilibrium equation and boundary conditions, we need more notations. We denote the unit outward normal on the boundary $\partial\Omega$ by $\underset{\sim}{n} = n_\alpha \boldsymbol{g}^\alpha$. Let the surface force density be $\underset{\sim}{p} = p^\alpha \boldsymbol{g}_\alpha$, and body force density be $\underset{\sim}{q} = q^\alpha \boldsymbol{g}_\alpha$. With these notations, the equilibrium equation can be written as

$$\sigma^{\alpha\beta}\|_\beta + q^\alpha = 0. \tag{2.2.6}$$

On the part of the domain boundary $\partial_T\Omega$, the surface force condition can be expressed as

$$\sigma^{\alpha\beta} n_\beta = p^\alpha. \tag{2.2.7}$$

On $\partial_D\Omega$, the body is clamped, so the condition is

$$v_\alpha = 0. \tag{2.2.8}$$

According to the linearized elasticity theory, the system of equations (2.2.4), (2.2.5), (2.2.6) together with the boundary conditions (2.2.7) and (2.2.8) uniquely determine the covariant components v^*_α of the displacement field of the elastic body arising in response to the applied forces and the prescribed clamping boundary condition. The stress distribution $\underset{\approx}{\sigma}^*$ is determined by giving its contravariant components $\sigma^{*\alpha\beta} = C^{\alpha\beta\lambda\gamma}\chi_{\lambda\gamma}(\underset{\sim}{v}^*)$.

The weak formulation of the plane strain elasticity equation is

$$\int_\Omega C^{\alpha\beta\lambda\gamma}\chi_{\lambda\gamma}(\underset{\sim}{v})\chi_{\alpha\beta}(\underset{\sim}{u}) = \int_\Omega q^\alpha u_\alpha + \int_{\partial_T\Omega} p^\alpha u_\alpha, \qquad \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\omega) \quad \forall\ \underset{\sim}{u} \in \underset{\sim}{H}^1_D(\omega), \tag{2.2.9}$$

in which $\underset{\sim}{H}^1_D(\omega)$ is the space of vector-valued functions that are square integrable and have square integrable first derivatives, and vanish on $\partial_D\omega$. It is clear that if q^α is in the dual space of $\underset{\sim}{H}^1_D(\omega)$, and p^α is in the dual space of the trace space $\underset{\sim}{H}^{1/2}_{00}(\partial_T\omega)$, the variational problem has a unique solution $\underset{\sim}{v}^* \in \underset{\sim}{H}^1_D(\omega)$.

A symmetric tensor field $\underset{\approx}{\sigma}$ is called a statically admissible stress field if it satisfies both the equilibrium equation (2.2.6) and the traction boundary condition (2.2.7). A vector field $\underset{\sim}{v} \in \underset{\sim}{H}^1(\omega)$ is called a kinematically admissible displacement field, if it satisfies the clamping boundary condition (2.2.8). For a statically admissible field $\underset{\approx}{\sigma}$ and a kinematically admissible field $\underset{\sim}{v}$, the following integration identity holds:

$$\begin{aligned} \int_\Omega A_{\alpha\beta\lambda\gamma}(\sigma^{\lambda\gamma} - \sigma^{*\lambda\gamma})(\sigma^{\alpha\beta} - \sigma^{*\alpha\beta}) \\ + \int_\Omega C^{\alpha\beta\lambda\gamma}[\chi_{\lambda\gamma}(\underset{\sim}{v}) - \chi_{\lambda\gamma}(\underset{\sim}{v}^*)][\chi_{\alpha\beta}(\underset{\sim}{v}) - \chi_{\alpha\beta}(\underset{\sim}{v}^*)] \\ = \int_\Omega [\sigma^{\alpha\beta} - C^{\alpha\beta\lambda\gamma}\chi_{\lambda\gamma}(\underset{\sim}{v})][A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma} - \chi_{\alpha\beta}(\underset{\sim}{v})]. \end{aligned} \tag{2.2.10}$$

This is the two energies principle, from which the minimum complementary energy principle and minimum potential energy principle easily follow. If we somehow obtain an approximate admissible stress field $\underset{\approx}{\sigma}$ and an approximate admissible displacement field $\underset{\sim}{v}$, then the two energies principle gives an *a posteriori* bound for the accuracies of $\underset{\approx}{\sigma}$ and $\underset{\sim}{v}$ in the energy norm by the norm of the residual of the constitutive equation. For the plane strain cylindrical shell problem, this identity will direct us to a model, and enable us to justify it.

2.2.3 Plane strain cylindrical shells

A plane strain cylindrical shell problem is a special plane strain elasticity problem, in which the cross section of the cylinder is a thin curvilinear rectangle. For simplicity, we assume that it is clamped on the two lateral sides and subjected to surface forces on its upper and lower surfaces, and loaded by a body force.

Let the middle curve $S \subset \mathbb{R}^2$ of the cross section be parameterized by its arc length through the mapping $\boldsymbol{\phi}$, i.e.,

$$S = \{\boldsymbol{\phi}(x) | x \in [0, L]\}.$$

With this parameterization, the tangent vector $\boldsymbol{a}_1 = \partial\boldsymbol{\phi}/\partial x$ is a unit vector at any point on S. At each point on S, we define the unit vector $\boldsymbol{a}_2$ that is orthogonal to the curve and lies on the same side of the curve for all points.

The cross section Ω^ϵ of the cylindrical shell, with middle curve S and thickness $2\,\epsilon$, occupies the region in $\mathbb{R}^2$ that is the image of the thin rectangle $\omega^\epsilon = [0, L] \times [-\,\epsilon, \epsilon]$ through the mapping

$$\boldsymbol{\Phi}(x, t) = \boldsymbol{\phi}(x) + t\boldsymbol{a}_2, \quad x \in [0, L],\ t \in [-\,\epsilon, \epsilon].$$

We assume that ϵ is small enough so that $\boldsymbol{\Phi}$ is injective. The pair of numbers (x, t) then furnishes curvilinear coordinates on the 2D domain Ω^ϵ, on which the plane strain shell problem is defined. We sometimes use the notation (x_1, x_2) to replace (x, t) for convenience. For brevity, the derivative ∂_x will be denoted by ∂. The boundary of Ω^ϵ is composed of the upper and lower sides $\Gamma_\pm = \boldsymbol{\Phi}((0, L) \times \{\pm\,\epsilon\})$ where the shell is subjected to surface forces, and the lateral sides $\Gamma_0 = \boldsymbol{\Phi}(\{0\} \times [-\,\epsilon, \epsilon])$ and $\Gamma_L = \boldsymbol{\Phi}(\{L\} \times [-\,\epsilon, \epsilon])$ where the shell is clamped.

The curvature of S at the point $\boldsymbol{\phi}(x)$ is defined by $b(x) = \boldsymbol{a}_2 \cdot \partial \boldsymbol{a}_1$. We denote the maximum absolute value of the curvature by $B = \max_{x \in [0,L]} |b(x)|$.

With the curvilinear coordinates defined on Ω^ϵ, the covariant basis vectors at (x,t) in Ω^ϵ are $\boldsymbol{g}_1 = (1 - bt)\boldsymbol{a}_1, \quad \boldsymbol{g}_2 = \boldsymbol{a}_2$. The covariant metric tensor $g_{\alpha\beta}$ is given by $g_{11} = (1 - bt)^2,\ g_{22} = 1,\ g_{12} = g_{21} = 0$, and the contravariant metric tensor $g^{\alpha\beta}$ is given by $g^{11} = 1/(1 - bt)^2,\ g^{22} = 1,\ g^{12} = g^{21} = 0$.

We denote the determinant of the covariant metric tensor by $g = \det(g_{\alpha\beta})$. Then the Jacobian of the transformation $\boldsymbol{\Phi}$ is $\sqrt{g} = 1 - bt$. Therefore,

$$\int_{\Omega^\epsilon} f \circ \boldsymbol{\Phi}^{-1} = \int_0^L \int_{-\epsilon}^{\epsilon} f(x,t)(1 - bt)\,dt\,dx \tag{2.2.11}$$

holds for all $f : \omega^\epsilon \to \mathbb{R}$. Often, we will simply write $\int_{\Omega^\epsilon} f$ instead of $\int_{\Omega^\epsilon} f \circ \boldsymbol{\Phi}^{-1}$.

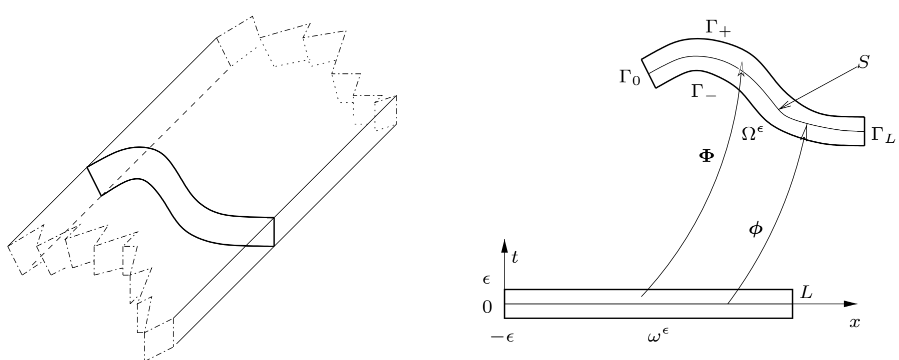


Fig. 2.1. A cylindrical shell and its cross section

The Christoffel symbols of this metric are

$$\Gamma^{*1}_{11} = \frac{-\partial bt}{1-bt}, \qquad \Gamma^{*1}_{12} = \frac{-b}{1-bt}, \qquad \Gamma^{*1}_{22} = 0,$$

$$\Gamma^{*2}_{11} = b(1-bt), \qquad \Gamma^{*2}_{12} = 0, \qquad \Gamma^{*2}_{22} = 0.$$

The geometric equation becomes

$$\begin{aligned} \chi_{11}(\underset{\sim}{v}) &= \partial v_1 + \frac{\partial bt}{1-bt} v_1 - b(1-bt) v_2, \quad \chi_{22}(\underset{\sim}{v}) = \partial_t v_2, \\ \chi_{12}(\underset{\sim}{v}) &= \chi_{21}(\underset{\sim}{v}) = \frac{1}{2}(\partial_t v_1 + \partial v_2) + \frac{b}{1-bt} v_1. \end{aligned} \tag{2.2.12}$$

The row divergence of a tensor field $\sigma^{\alpha\beta}$, by (2.2.3), has the expression

$$\begin{aligned} \sigma^{1\beta}\|_\beta &= \partial\sigma^{11} + \partial_t \sigma^{12} - 2\frac{\partial bt}{1-bt}\sigma^{11} - 3\frac{b}{1-bt}\sigma^{12}, \\ \sigma^{2\beta}\|_\beta &= \partial\sigma^{12} + \partial_t\sigma^{22} + b(1-bt)\sigma^{11} - \frac{\partial bt}{1-bt}\sigma^{12} - \frac{b}{1-bt}\sigma^{22}. \end{aligned} \tag{2.2.13}$$

Let the surface force densities on $\Gamma_\pm$ be $\underset{\sim}{p}_\pm = p^\alpha_\pm \boldsymbol{g}_\alpha$, the body force density be $\underset{\sim}{q} = q^\alpha \boldsymbol{g}_\alpha$. The equilibrium equation is

$$\sigma^{\alpha\beta}\|_\beta + q^\alpha = 0. \tag{2.2.14}$$

The traction boundary conditions on $\Gamma_\pm$ expressed in terms of the contravariant components of a stress field $\underset{\approx}{\sigma}$ read

$$\sigma^{12}(\,\cdot\,,\epsilon) = p^1_+, \quad \sigma^{12}(\,\cdot\,,-\epsilon) = -p^1_-, \quad \sigma^{22}(\,\cdot\,,\epsilon) = p^2_+, \quad \sigma^{22}(\,\cdot\,,-\epsilon) = -p^2_-. \tag{2.2.15}$$

According to the definition, a stress field $\underset{\approx}{\sigma}$ is statically admissible if both the equations (2.2.14) and (2.2.15) are satisfied by its contravariant components.

The clamping boundary condition imposed on an admissible displacement field $\underset{\sim}{v}(x,t)$ is simply

$$v_1(0,\,\cdot\,) = v_1(L,\,\cdot\,) = v_2(0,\,\cdot\,) = v_2(L,\,\cdot\,) = 0. \tag{2.2.16}$$

2.2.4 Rescaled stress and displacement components

To simplify the calculation, we introduce the rescaled components $\tilde{\sigma}^{\alpha\beta}$ for a stress tensor $\sigma^{\alpha\beta}$ by

$$\tilde{\sigma}^{11} = (1-bt)^2\sigma^{11}, \quad \tilde{\sigma}^{12} = (1-bt)\sigma^{12}, \quad \tilde{\sigma}^{22} = (1-bt)\sigma^{22}. \tag{2.2.17}$$

Then

$$\begin{aligned} \sigma^{1\beta}\|_\beta &= \frac{1}{(1-bt)^2}[\partial\tilde{\sigma}^{11} + (1-bt)\partial_t\tilde{\sigma}^{12} - 2b\tilde{\sigma}^{12}], \\ \sigma^{2\beta}\|_\beta &= \frac{1}{1-bt}[\partial\tilde{\sigma}^{12} + \partial_t\tilde{\sigma}^{22} + b\tilde{\sigma}^{11}], \end{aligned} \tag{2.2.18}$$

which is noticeably simpler than (2.2.13).

In these curvilinear coordinates, and in terms of the rescaled stress components, the constitutive equation

$$\chi_{\alpha\beta} = A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma}$$

takes the form

$$\chi_{11} = \frac{2\mu+\lambda}{4\mu(\mu+\lambda)}(1-bt)^2\tilde{\sigma}^{11} - \frac{\lambda}{4\mu(\mu+\lambda)}(1-bt)\tilde{\sigma}^{22},$$

$$\chi_{12} = \chi_{21} = \frac{1}{2\mu}(1-bt)\tilde{\sigma}^{12}, \tag{2.2.19}$$

$$\chi_{22} = \frac{2\mu+\lambda}{4\mu(\mu+\lambda)}\frac{1}{1-bt}\tilde{\sigma}^{22} - \frac{\lambda}{4\mu(\mu+\lambda)}\tilde{\sigma}^{11}.$$

For consistency with the rescaled stress components, we introduce the rescaled components $\tilde{q}^{\alpha}$ for the body force density and rescaled components $\tilde{p}^{\alpha}$ for the surface force density.

For the body force density, we define the rescaled components by

$$\underset{\sim}{q} = q^{\alpha}\boldsymbol{g}_{\alpha} = \tilde{q}^{\alpha}\frac{1}{1-bt}\boldsymbol{a}_{\alpha}. \tag{2.2.20}$$

In components, we have $\tilde{q}^1 = (1-bt)^2 q^1$ and $\tilde{q}^2 = (1-bt)q^2$. The rescaled components account the area change in the transverse direction of the cross section and more explicitly reflect the variation of the body force density in that direction. We define the components of the transverse average and moment of the body force density by

$$q_a^{\alpha} = \frac{1}{2\,\epsilon}\int_{-\,\epsilon}^{\epsilon}\underset{\sim}{q}\cdot\boldsymbol{a}^{\alpha}dt, \quad q_m^{\alpha} = \frac{3}{2\,\epsilon^3}\int_{-\,\epsilon}^{\epsilon}t\underset{\sim}{q}\cdot\boldsymbol{a}^{\alpha}dt. \tag{2.2.21}$$

In the following, we assume the body force density changes linearly in t, or equivalently, $\underset{\sim}{q} = (q_a^{\alpha} + tq_m^{\alpha})\boldsymbol{a}_{\alpha}$. Under this assumption, the rescaled components are quadratic polynomials in t, and we have $\tilde{q}^\alpha = q_0^\alpha + tq_1^\alpha + t^2 q_2^\alpha$, with $q_0^\alpha = q_a^\alpha$, $q_1^\alpha = q_m^\alpha - bq_a^\alpha$, and $q_2^\alpha = -bq_m^\alpha$.

The ensuing calculations can be carried through if $\tilde{q}^\alpha$ are arbitrary quadratic polynomials in t. Without this restriction, we cannot apply the two energies principle directly. For a general body force density, the convergence of the model can be proved under some restriction on the transverse variation of the body force density. This issue will be addressed in the general shell theory.

For the surface force density $\underset{\sim}{p}_\pm$, we introduce the rescaled components $\tilde{p}_\pm^\alpha$ by

$$\underset{\sim}{p}_+ = p_+^\alpha \boldsymbol{g}_\alpha = \tilde{p}_+^\alpha \frac{1}{1 - b\,\epsilon} \boldsymbol{g}_\alpha, \quad \underset{\sim}{p}_- = p_-^\alpha \boldsymbol{g}_\alpha = \tilde{p}_-^\alpha \frac{1}{1 + b\,\epsilon} \boldsymbol{g}_\alpha. \tag{2.2.22}$$

The rescaled components account the length differences of the upper and lower curves of the shell cross section from middle curve. In terms of the rescaled surface force components, we define

$$p_o^1 = \frac{\tilde{p}_+^1 - \tilde{p}_-^1}{2}, \quad p_o^2 = \frac{\tilde{p}_+^2 - \tilde{p}_-^2}{2}, \quad p_e^1 = \frac{\tilde{p}_+^1 + \tilde{p}_-^1}{2\,\epsilon}, \quad p_e^2 = \frac{\tilde{p}_+^2 + \tilde{p}_-^2}{2\,\epsilon}, \tag{2.2.23}$$

which are the odd and weighted even parts of the upper and lower surface forces.

In terms of the rescaled stress components $\tilde{\sigma}^{\alpha\beta}$ and the rescaled applied force components, the equilibrium equation (2.2.14) and the surface force condition (2.2.15) can be written as

$$\begin{aligned} &\partial\tilde{\sigma}^{11} + (1 - bt)\partial_t \tilde{\sigma}^{12} - 2b\tilde{\sigma}^{12} + \tilde{q}^1 = 0, \\ &\partial\tilde{\sigma}^{12} + \partial_t \tilde{\sigma}^{22} + b\tilde{\sigma}^{11} + \tilde{q}^2 = 0 \end{aligned} \tag{2.2.24}$$

and

$$\tilde{\sigma}^{12}(\,\cdot\,,\pm\epsilon) = p_o^1 \pm \epsilon\, p_e^1, \quad \tilde{\sigma}^{22}(\,\cdot\,,\pm\epsilon) = p_o^2 \pm \epsilon\, p_e^2. \tag{2.2.25}$$

We introduce the rescaled displacement components $\tilde{v}_\alpha$ for the displacement vector $\underset{\sim}{v}$ by expressing it as the combination of basis vectors on the middle curve, i.e., $\underset{\sim}{v} = v_\alpha \boldsymbol{g}^\alpha = \tilde{v}_\alpha \boldsymbol{a}^\alpha$, or equivalently, $v_1 = (1-bt)\tilde{v}_1, \quad v_2 = \tilde{v}_2$. In terms of the rescaled components $\tilde{v}_\alpha$, by using (2.2.12), the geometric equation becomes

$$\begin{aligned} \chi_{11}(\underset{\sim}{v}) &= (1-bt)(\partial\tilde{v}_1 - b\tilde{v}_2), \quad \chi_{22}(\underset{\sim}{v}) = \partial_t\tilde{v}_2, \\ \chi_{12}(\underset{\sim}{v}) &= \chi_{21}(\underset{\sim}{v}) = \frac{1}{2}[b\tilde{v}_1 + \partial\tilde{v}_2 + (1-bt)\partial_t\tilde{v}_1]. \end{aligned} \tag{2.2.26}$$

And the clamping boundary condition is

$$\tilde{v}_\alpha(0,\,\cdot\,) = \tilde{v}_\alpha(L,\,\cdot\,) = 0. \tag{2.2.27}$$

In summary, in terms of the rescaled components, the elasticity problem seeks displacement components $\tilde{v}_\alpha$ and stress components $\tilde{\sigma}^{\alpha\beta}$ satisfying the constitutive equation (2.2.19), the equilibrium equation (2.2.24), the geometric equation (2.2.26) and the boundary conditions (2.2.25) and (2.2.27).

2.3 The shell model

Our shell model is a 1D variational problem defined on the space $H = [H_0^1(0,L)]^3$. The solution of the model is composed of three single variable functions that approximately describe the shell displacement arising in response to the applied forces and boundary conditions. For any $(\theta, u, w) \in H$, we define

$$\gamma(u,w) = \partial u - bw, \quad \rho(\theta,u,w) = \partial\theta + b(\partial u - bw), \quad \tau(\theta,u,w) = \theta + \partial w + bu, \quad (2.3.1)$$

which give the membrane strain, flexural strain and shear strain engendered by the displacement functions (θ, u, w).

The model reads: Find $(\theta^\epsilon, u^\epsilon, w^\epsilon) \in H$, such that

$$\begin{aligned} &\frac{1}{3}\epsilon^2(2\mu + \lambda^\star)\int_0^L \rho(\theta^\epsilon, u^\epsilon, w^\epsilon)\rho(\phi, y, z)dx \\ &\quad + (2\mu + \lambda^\star)\int_0^L \gamma(u^\epsilon, w^\epsilon)\gamma(y,z)dx + \frac{5}{6}\mu\int_0^L \tau(\theta^\epsilon, u^\epsilon, w^\epsilon)\tau(\phi, y, z)dx \\ &\quad = \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\phi, y, z)\rangle \quad \forall(\phi, y, z) \in H, \quad (2.3.2) \end{aligned}$$

in which

$$\lambda^\star = \frac{2\mu\lambda}{2\mu + \lambda},$$

and the resultant loading functionals are given by

$$\begin{aligned} \langle \boldsymbol{f}_0, (\phi, y, z)\rangle = &\frac{5}{6}\int_0^L p_o^1 \tau(\phi, y, z)dx - \frac{\lambda}{2\mu + \lambda}\int_0^L p_o^2 \gamma(y,z) \\ &+ \int_0^L [(p_e^1 + q_a^1 - 2bp_o^1)y + (p_e^2 + q_a^2 + \partial p_o^1)z]dx \quad (2.3.3) \end{aligned}$$

and

$$\langle \boldsymbol{f}_1, (\phi, y, z)\rangle = -\frac{1}{3}\int_0^L [(bq_a^1 + 3bp_e^1 - q_m^1)\phi + bq_m^1 y + bq_m^2 z]dx$$

$$- \frac{\lambda}{3(2\mu+\lambda)} \int_0^L (p_e^2 + bp_o^2)\rho(\phi, y, z)dx - \frac{1}{6}\int_0^L bp_e^1\tau(\phi, y, z)dx. \quad (2.3.4)$$

The bilinear form in the left hand side of the variational formulation of the model (2.3.2) is uniformly elliptic in the space $H = [H_0^1(0, L)]^3$. This conclusion follows from the following theorem.

THEOREM 2.3.1. *The equivalency*

$$\|\rho(\theta, u, w)\|_{L_2(0,L)} + \|\gamma(u, w)\|_{L_2(0,L)} + \|\tau(\theta, u, w)\|_{L_2(0,L)} \simeq \|(\theta, u, w)\|_H \quad (2.3.5)$$

holds for all $(\theta, u, w) \in H = [H_0^1(0, L)]^3$. *Here* ρ, γ *and* τ *are the strain operators defined in (2.3.1).*

To prove this result, we need Peetre's lemma.

LEMMA 2.3.2. *Let* X, Y_1, Y_2 *be Hilbert spaces, and let* $A_1 : X \to Y_1$ *and* $A_2 : X \to Y_2$ *be bounded linear operators with* A_1 *injective and* A_2 *compact. If there exists a constant* $c > 0$ *such that*

$$\|x\|_X \leq c(\|A_1 x\|_{Y_1} + \|A_2 x\|_{Y_2}) \quad \forall x \in X,$$

then there exists a constant $c' > 0$ *such that*

$$\|x\|_X \leq c'\|A_1 x\|_{Y_1} \quad \forall x \in X.$$

For a proof of this lemma, see [28]. We give the proof of the theorem.

Proof of Theorem 2.3.1. The upper bound of the left hand side is obvious. For the lower bound, we first see that

$$\|\rho(\theta,u,w)\|_{L_2(0,L)} + (1+B)\|\gamma(u,w)\|_{L_2(0,L)} + \|\tau(\theta,u,w)\|_{L_2(0,L)} \geq \|\partial\theta\|_{L_2(0,L)} + \|\partial u - bw\|_{L_2(0,L)} + \|\partial w + \theta + bu\|_{L_2(0,L)}.$$

We consider the operators A_1 and A_2 from H to $[L_2(0,L)]^3$ defined by,

$$A_1(\theta,u,w) = (\partial\theta, \partial u - bw, \partial w + \theta + bu), \quad A_2(\theta,u,w) = (0, bw, \theta + bu), \ \forall\ (\theta,u,w) \in H.$$

The operator A_1 is injective, since if $(\theta,u,w) \in \ker A_1$, then $\theta = 0$, $\partial u - bw = 0$ and $\partial w + bu = 0$, so $u\partial u + w\partial w = 0$, therefore, $u^2 + w^2 =$ constant. Since u and w vanish on the end points of the interval, we must have $u = w = 0$. The operator A_2 is obviously compact. The statement follows from Lemma 2.3.2. □

Theorem 2.3.1 shows that if the resultant loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$ is in the dual space of H, the model problem is uniquely solvable.

Remark 2.3.1. *The requirement $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1 \in H^*$ can be met, if, say, the the applied force functions are square integrable. To prove the convergence, we will need to assume the tangential surface forces $\tilde{p}^1_\pm \in H^1(0,L)$. To prove the best possible convergence rate, we will further need to assume the normal surface forces $\tilde{p}^2_\pm \in H^1(0,L)$. Henceforth, we will assume that*

$$\tilde{p}^\alpha_\pm \in H^1(0,L),\ q^\alpha_a,\ q^\alpha_m \in L_2(0,L). \tag{2.3.6}$$

This model is slightly different from that of Naghdi's in the following aspects:

1. There is a shear correction factor 5/6. The best value for this factor is an unresolved issue in shell theories. For the special case of plate, the value 5/6 is usually accepted as the best. We will see that in the flexural case, the problem is not sensitive to this value. In the case of membrane–shear, if this factor is changed, there must be a corresponding change in the resultant loading functional, otherwise a poor choice of the factor may lead to divergence of the model.

2. The expression for flexural strain is $\partial\theta + b(\partial u - bw)$ while in the classical Naghdi model it is $\partial\theta - b(\partial u - bw)$. This change of the flexural strain operator rooted in our derivation of the model, in which, the dimensionally reduced constitutive equation was derived by roughly minimizing constitutive residual. Our choice leads to a smaller constitutive residual. See Remark 2.4.1. Aother evidence favoring this change is provided by modeling a semi-circular cylindrical shell, in which this change is simply a consequence of more accurate integrations in the transverse direction in the process of classical Naghdi model derivation.

3. The resultant loading functional contains more information than is normally retained in the Naghdi model. The model convergence and convergence rate in the relative energy norm can be proved if only $\boldsymbol{f}_0$ is kept in the loading functional. See Section 7.1.

2.4 Reconstruction of the stress and displacement fields

From the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon) \in [H_0^1(0,L)]^3$, we can rebuild a statically admissible stress field by explicitly giving its contravariant components, and a kinematically admissible displacement field by giving its covariant components. We will prove the convergence of both the reconstructed stress field and displacement field to the actual fields determined from the 2D elasticity equations in the shell. The convergence will be proved by using the two energies principle. To this end, we need to compute the constitutive residual. We will see that the residual is formally small. Knowledge of the behavior of the model solution will be necessary for a rigorous proof of the convergence.

2.4.1 Reconstruction of the statically admissible stress field

For brevity, we denote the flexural, membrane, and shear strains engendered by the model solution by

$$\rho^\epsilon = \rho(\theta^\epsilon, u^\epsilon, w^\epsilon), \quad \gamma^\epsilon = \gamma(u^\epsilon, w^\epsilon), \quad \tau^\epsilon = \tau(\theta^\epsilon, u^\epsilon, w^\epsilon).$$

We define three single variable functions $\sigma_1^{11}, \sigma_0^{11}$, and σ_0^{12} by

$$\begin{aligned}
\sigma_1^{11} &= (2\mu + \lambda^\star)\rho^\epsilon + \frac{\lambda}{2\mu+\lambda}(p_e^2 + b p_o^2), \\
\sigma_0^{11} &= \frac{1}{3} b\,\epsilon^2\,\sigma_1^{11} + (2\mu+\lambda^\star)\gamma^\epsilon + \frac{\lambda}{2\mu+\lambda} p_o^2, \\
\sigma_0^{12} &= \frac{5}{4}\mu\tau^\epsilon - \frac{5}{4}p_o^1 + \frac{1}{4} b\epsilon^2 p_e^1,
\end{aligned} \tag{2.4.1}$$

which furnish the principal part of the statically admissible stress field. It is straightforward to verify that these functions satisfy the following equations:

$$\begin{aligned}
\frac{1}{3}\,\epsilon^2\,\partial\sigma_1^{11} - \frac{2}{3}\sigma_0^{12} &= \epsilon^2\,bp_e^1 + \frac{1}{3}\,\epsilon^2(bq_a^1 - q_m^1),\\
\partial\sigma_0^{11} - \frac{2}{3}b\sigma_0^{12} &= 2bp_o^1 - p_e^1 - q_a^1 + \frac{1}{3}\,\epsilon^2\,bq_m^1,\\
b\sigma_0^{11} + \frac{2}{3}\partial\sigma_0^{12} &= -p_e^2 - \partial p_o^1 - q_a^2 + \frac{1}{3}\,\epsilon^2\,bq_m^2.
\end{aligned} \tag{2.4.2}$$

Actually, by substituting (2.4.1) into (2.4.2), we will get a system of three second order ordinary differential equations, which is just the differential form of the variational model equation (2.3.2). Obviously, the three principal stress functions are in $L_2(0,L)$. Furthermore, the equations in (2.4.2) clearly show that these three functions are in $H^1(0,L)$.

To complete the construction of a statically admissible stress field, we also need three supplementary functions σ_2^{11}, σ_0^{22}, and σ_1^{22}. They are defined by

$$\begin{aligned}
\partial\sigma_2^{11} &= -4b\sigma_0^{12} + \epsilon^2\,bq_m^1,\\
\sigma_0^{22} &= \frac{1}{2}\epsilon^2(b\sigma_1^{11} + \partial p_e^1 + q_m^2 - bq_a^2),\\
\sigma_1^{22} &= \frac{1}{2}\epsilon(\frac{2}{3}b\sigma_2^{11} + b\sigma_0^{11} + p_e^2 + \partial p_o^1 + q_a^2 - \epsilon^2\,bq_m^2).
\end{aligned} \tag{2.4.3}$$

Note that the first equation in (2.4.3) only gives $\partial\sigma_2^{11}$, so σ_2^{11} is determined up to an arbitrary additive constant. We fix a particular solution by requiring $\int_0^L \sigma_2^{11} = 0$. Then

$$\|\sigma_2^{11}\|_{H^1(0,L)} \lesssim B(\|\sigma_0^{12}\|_0 + \epsilon^2\,\|q_m^1\|_0). \tag{2.4.4}$$

With these six functions determined, the rescaled stress components $\tilde{\sigma}^{\alpha\beta}$ then are explicitly defined by

$$\tilde{\sigma}^{11} = \sigma_0^{11} + t\sigma_1^{11} + r(t)\sigma_2^{11},$$
$$\tilde{\sigma}^{12} = \tilde{\sigma}^{21} = p_o^1 + tp_e^1 + q(t)\sigma_0^{12}, \tag{2.4.5}$$
$$\tilde{\sigma}^{22} = p_o^2 + tp_e^2 + q(t)\sigma_0^{22} + s(t)\sigma_1^{22},$$

where

$$r(t) = \frac{t^2}{\epsilon^2} - \frac{1}{3}, \quad q(t) = 1 - \frac{t^2}{\epsilon^2}, \quad s(t) = \frac{t}{\epsilon}(1 - \frac{t^2}{\epsilon^2}). \tag{2.4.6}$$

Note that r is an even function of t and has zero integral over the interval $[-\epsilon, \epsilon]$, and $q(\pm\epsilon) = s(\pm\epsilon) = 0$. Following classical terminology, we will call σ_0^{11} the resultant membrane stress, σ_1^{11} the first membrane stress moment, and σ_2^{11} the second membrane stress moment. The function σ_0^{12} is responsible for the quadratic distribution of the rescaled shear stress in the transverse direction and will be shown to be a higher order term. The two functions σ_0^{22} and σ_1^{22} enrich the variation of the normal stress in the transverse direction.

With this choice of the rescaled stress components, the surface traction condition (2.2.25) is precisely satisfied. Combining the six equations in (2.4.2) and (2.4.3) and the definition (2.4.5), we can verify that the equilibrium equation (2.2.24) is precisely satisfied. Therefore, by the relation between the rescaled components and the contravariant components (2.2.17), we get the contravariant components $\sigma^{\alpha\beta}$ of a statically admissible stress field $\underset{\approx}{\sigma}$.

$$\sigma^{11} = \frac{1}{(1-bt)^2}[\sigma_0^{11} + t\sigma_1^{11} + r(t)\sigma_2^{11}],$$

$$\sigma^{12} = \sigma^{21} = \frac{1}{1-bt}[p_o^1 + tp_e^1 + q(t)\sigma_0^{12}], \tag{2.4.7}$$

$$\sigma^{22} = \frac{1}{1-bt}[p_o^2 + tp_e^2 + q(t)\sigma_0^{22} + s(t)\sigma_1^{22}].$$

2.4.2 Reconstruction of the kinematically admissible displacement field

The rescaled components of the displacement field are defined by

$$\tilde{v}_1 = u^\epsilon + t\theta^\epsilon, \quad \tilde{v}_2 = w^\epsilon + tw_1 + t^2 w_2. \tag{2.4.8}$$

Here, $w_1 \in H_0^1(0,L)$ and $w_2 \in H_0^1(0,L)$ are two correction functions defined as solutions of the following equations.

$$\epsilon^2(\partial w_1, \partial v)_{L_2(0,L)} + (w_1, v)_{L_2(0,L)} = (\frac{1}{2\mu+\lambda^\star}[p_o^2 - \frac{\lambda}{2\mu+\lambda}\sigma_0^{11}], v)_{L_2(0,L)} \quad \forall\, v \in H_0^1(0,L) \tag{2.4.9}$$

and

$$\epsilon^2(\partial w_2, \partial v)_{L_2(0,L)} + (w_2, v)_{L_2(0,L)} = (\frac{1}{2(2\mu+\lambda^\star)}[p_e^2 - \frac{\lambda}{2\mu+\lambda}\sigma_1^{11}], v)_{L_2(0,L)} \quad \forall\, v \in H_0^1(0,L). \tag{2.4.10}$$

The clamping boundary condition (2.2.27) is obviously satisfied. Note that this correction does not affect the middle curve displacement. So the basic pattern of the shell deformation is already well captured by the model solution. The covariant components of the kinematically admissible displacement field $\underset{\sim}{v}$ are

$$v_1 = (1 - bt)(u^\epsilon + t\theta^\epsilon), \quad v_2 = w^\epsilon + tw_1 + t^2 w_2. \tag{2.4.11}$$

These components are in $H^1(\omega^\epsilon)$, and satisfy the requirement of the two energies principle.

2.4.3 Constitutive residual

We denote the residual of the constitutive equation by $\varrho_{\alpha\beta} = A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma} - \chi_{\alpha\beta}(\underset{\sim}{v})$, in which $\sigma^{\alpha\beta}$ and v_α are the components of the admissible stress and displacement fields constructed from the model solution in the previous subsections.

By the formulae (2.2.26), we have

$$\begin{aligned}
\chi_{11}(\underset{\sim}{v}) &= (1 - bt)(\partial u^\epsilon + t\partial\theta^\epsilon - bw^\epsilon - btw_1 - bt^2 w_2) \\
&= \gamma^\epsilon + t\rho^\epsilon - 2bt\gamma^\epsilon - b(1 - bt)(tw_1 + t^2 w_2) - bt^2\partial\theta^\epsilon, \\
\chi_{12}(\underset{\sim}{v}) &= \chi_{21}(\underset{\sim}{v}) = \frac{1}{2}(\theta^\epsilon + \partial w^\epsilon + bu^\epsilon + t\partial w_1 + t^2\partial w_2) \\
&= \frac{1}{2}\tau^\epsilon + \frac{1}{2}(t\partial w_1 + t^2\partial w_2), \\
\chi_{22}(\underset{\sim}{v}) &= w_1 + 2tw_2.
\end{aligned} \tag{2.4.12}$$

By the formulae (2.2.19), the definitions (2.4.1) and (2.4.5), and the identity $(2\mu+\lambda)/[4\mu(\mu+\lambda)] = 1/(2\mu+\lambda^\star)$, we have

$$\begin{aligned}
A_{11\lambda\gamma}\sigma^{\lambda\gamma} &= \gamma^\epsilon + t\rho^\epsilon - 2bt\gamma^\epsilon \\
&+ \frac{1}{2\mu+\lambda^\star}\{b^2t^2[\sigma_0^{11} + t\sigma_1^{11} + r(t)\sigma_2^{11}] + [\frac{1}{3}b\,\epsilon^2(1-2bt) - 2bt^2]\sigma_1^{11}\} \\
&- \frac{\lambda}{4\mu(\mu+\lambda)}\{(1-bt)[q(t)\sigma_0^{22} + s(t)\sigma_1^{22}] - bt^2p_e^2\} \\
&+ \frac{1}{2\mu+\lambda^\star}(1-2bt)r(t)\sigma_2^{11}, \qquad (2.4.13) \\
A_{12\lambda\gamma}\sigma^{\lambda\gamma} &= \frac{1}{2\mu}(1-bt)[p_o^1 + tp_e^1 + q(t)\sigma_0^{12}], \\
A_{22\lambda\gamma}\sigma^{\lambda\gamma} &= \frac{1}{2\mu+\lambda^\star}(p_o^2 - \frac{\lambda}{2\mu+\lambda}\sigma_0^{11}) + t\frac{1}{2\mu+\lambda^\star}(p_e^2 - \frac{\lambda}{2\mu+\lambda}\sigma_1^{11}) \\
&+ \frac{1}{2\mu+\lambda^\star}\{q(t)\sigma_0^{22} + s(t)\sigma_1^{22} + \frac{bt}{1-bt}[p_o^2 + tp_e^2 + q(t)\sigma_0^{22} + s(t)\sigma_1^{22}]\} \\
&- \frac{\lambda}{4\mu(\mu+\lambda)}r(t)\sigma_2^{11}.
\end{aligned}$$

Subtracting (2.4.12) from (2.4.13), we obtain the following expressions for the constitutive residual:

$$\begin{aligned}
\varrho_{11} &= \frac{1}{2\mu+\lambda^\star}\{b^2t^2[\sigma_0^{11} + t\sigma_1^{11} + r(t)\sigma_2^{11}] + [\frac{1}{3}b\,\epsilon^2(1-2bt) - 2bt^2]\sigma_1^{11}\} \\
&- \frac{\lambda}{4\mu(\mu+\lambda)}\{(1-bt)[q(t)\sigma_0^{22} + s(t)\sigma_1^{22}] - bt^2p_e^2\} \\
&+ b(1-bt)(tw_1 + t^2w_2) + bt^2\partial\theta^\epsilon \\
&+ \frac{1}{2\mu+\lambda^\star}(1-2bt)r(t)\sigma_2^{11}, \qquad (2.4.14)
\end{aligned}$$

$$\varrho_{12} = \frac{1}{2\mu}[\frac{5}{4}q(t) - 1](\mu\tau^\epsilon - p_o^1) - \frac{1}{2}(t\partial w_1 + t^2\partial w_2)$$

$$+\frac{1}{2\mu}[t+\frac{1}{4}q(t)b\,\epsilon^2]p_e^1-\frac{1}{2\mu}bt[p_o^1+tp_e^1+q(t)\sigma_0^{12}] \quad (2.4.15)$$

$$\begin{aligned}\varrho_{22}=&[\frac{1}{2\mu+\lambda^\star}(p_o^2-\frac{\lambda}{2\mu+\lambda}\sigma_0^{11})-w_1]+t[\frac{1}{2\mu+\lambda^\star}(p_e^2-\frac{\lambda}{2\mu+\lambda}\sigma_1^{11})-2w_2]\\&+\frac{1}{2\mu+\lambda^\star}\{q(t)\sigma_0^{22}+s(t)\sigma_1^{22}+\frac{bt}{1-bt}[p_o^2+tp_e^2+q(t)\sigma_0^{22}+s(t)\sigma_1^{22}]\}\\&-\frac{\lambda}{4\mu(\mu+\lambda)}r(t)\sigma_2^{11}.\end{aligned} \quad (2.4.16)$$

REMARK 2.4.1. *If we had not made the sign change in the flexural strain* $\rho(\theta,u,w)$ *discussed earlier, there would be an additional term* $2bt\gamma(u^\epsilon,w^\epsilon)$ *in the residual* ϱ_{11}. *Our variant does make the residual smaller, at least formally.*

Formally, most of the terms in the above residual expressions contain a factor of the form ϵ, t or smaller (recall that σ_0^{22} and σ_1^{22} have a small factor in their own expressions (2.4.3)). In the expression of ϱ_{11}, the only term not formally small is the last one, whose magnitude is determined by that of σ_2^{11}. The big term in the expression of ϱ_{12} is in the first one, which is determined by

$$\mu\tau^\epsilon-p_o^1. \quad (2.4.17)$$

This term is also the dominant part in the expression of σ_0^{12}, see (2.4.1). We will prove that $\mu\tau^\epsilon-p_o^1$ is indeed small. Therefore, σ_0^{12} is small, and by (2.4.4), so is σ_2^{11}.

The definitions (2.4.9) and (2.4.10) of the correction functions w_1 and w_2 were made to minimize the first two terms in the expression of ϱ_{22}, at the same time, they minimize the two terms $t\partial w_1$ and $t^2\partial w_2$ in the expression of ϱ_{12}. Therefore, we shall be able to show that ϱ_{22} is small as well.

2.5 Justification

The formal observations we made in the previous section do not furnish a rigorous justification, since the applied forces and the model solution may depend on the the shell thickness. To prove the convergence, we need to make some assumptions on the applied loads, and get a good grasp of the behavior of the model solution when the shell thickness tends to zero. Since we wish to bound the relative error, in addition to the upper bound that can be determined from the constitutive residual, we need to have a lower bound on the model solution.

2.5.1 Assumption on the applied forces

Henceforth, we assume that all the applied force functions explicitly involved in the resultant loading functional of the model are independent of ϵ, i.e., the single variable functions

$$p_o^\alpha, p_e^\alpha, q_a^\alpha, \text{ and } q_m^\alpha \text{ are independent of } \epsilon. \tag{2.5.1}$$

This assumption is different from the usual assumption adopted in asymptotic theories, according to which, the functions $\epsilon^{-1} p_o^\alpha$, rather than p_o^α themselves, should have been assumed to be independent of ϵ. Our assumption on p_e^α, q_a^α and q_m^α is the same as the usual one. This different assumption will reveal the potential advantages of the Naghdi-type model over the Koiter-type model. The convergence theorem can also be proved under the usual assumption on the applied forces, but it can be proved that the difference between the two types of models then is negligible.

2.5.2 An abstract theory

Under the assumption (2.5.1) on the applied forces, the model (2.3.2) is an ϵ-dependent variational problem fitting into the abstract problem that we shall discuss in Chapter 3, cf., (3.2.2). The following convergence bounds (2.5.4) and (2.5.7) easily follow from Theorem 3.3.1.

Let U, V, and H be Hilbert spaces, $A : H \to U$ a bounded linear operator, and $B : H \to V$ a bounded linear continuous surjection. We assume that

$$\|Au\|_U + \|Bu\|_V \simeq \|u\|_H \quad \forall\, u \in H. \tag{2.5.2}$$

For any $f_0, f_1 \in H^*$ and $f_0 \neq 0$, we consider the variational problem

$$\begin{gathered} \epsilon^2 (Au, Av)_U + (Bu, Bv)_V = \langle f_0 + \epsilon^2 f_1, v \rangle, \\ u \in H, \ \forall v \in H. \end{gathered} \tag{2.5.3}$$

It is obvious that under the equivalency assumption (2.5.2), this variational problem has a unique solution $u^\epsilon \in H$ that is dependent on ϵ. When $\epsilon \to 0$, the behavior of the solution u^ϵ is drastically different depending on whether $f_0|_{\ker B}$ is nonzero or not. As we shall see, in the former case, the solution u^ϵ blows up at the rate of $O(\epsilon^{-2})$, while in the latter case u^ϵ tends to a finite limit.

For the first case, to get more accurate description of the behavior of the solution, we rescale the problem by assuming $f_0 = \epsilon^2 F_0$ and $f_1 = \epsilon^2 F_1$ with $F_0, F_1 \in H$ independent of ϵ. Under this assumption, we have the convergence estimate

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V \lesssim \epsilon \|F_0\|_{H^*} + \epsilon^2 \|F_1\|_{H^*}, \tag{2.5.4}$$

in which $u^0 \in \ker B$ is independent of ϵ and is the solution of the limit problem

$$(Au^0, Av)_U = \langle F_0, v\rangle \quad \forall\, v \in \ker B. \tag{2.5.5}$$

Since $F_0|_{\ker B} \neq 0$, we must have $Au^0 \neq 0$.

For the second case, since $f_0 \in (\ker B)^a$ (the annihilator of $\ker B$) and B is surjective, there exists a unique $\zeta^0 \in V$, such that

$$\langle f_0, v\rangle = (\zeta^0, Bv)_V \quad \forall\, v \in H. \tag{2.5.6}$$

In this case, there exists a unique $u^0 \in H$ such that $Bu^0 = \zeta^0$, and we have the convergence estimate

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1} \|Bu^\epsilon - \zeta^0\|_V \lesssim \epsilon(\|f_0\|_{H^*} + \|f_1\|_{H^*}). \tag{2.5.7}$$

It can be shown that the limit u^0 can be determined as $u^0 = u_0^0 + u_1^0$. Here $(Au_0^0, Av)_U + (Bu_0^0, Bv)_V = 0 \;\; \forall v \in \ker B$, i.e., u_0^0 is in the orthogonal complement of $\ker B$ in H with respect to the inner product $(A\,\cdot\,, A\,\cdot\,)_U + (B\,\cdot\,, B\,\cdot\,)_V$ that, due to the equivalency assumption (2.5.2), is equivalent to the original inner product of H. And $u_1^0 \in \ker B$ is the solution of the limit problem corresponding to f_1,

$$(Au_1^0, Av)_U = \langle f_1, v\rangle \quad \forall\, v \in \ker B. \tag{2.5.8}$$

Since $f_0 \neq 0$, we have $\zeta^0 \neq 0$.

2.5.3 Asymptotic behavior of the model solution

To fit the model problem (2.3.2) in the abstract framework (2.5.3), we introduce the following Hilbert spaces,

$$H = [H_0^1(0,L)]^3, \quad U = L_2(0,L), \quad V = [L_2(0,L)]^2.$$

The inner product in H is the usual one. The inner products in U and V will be changed slightly and equivalently. For $\rho_1, \rho_2 \in U$, we define

$$(\rho_1, \rho_2)_U = \frac{1}{3}(2\mu + \lambda^\star)(\rho_1, \rho_2)_{L_2(0,L)}$$

and for $[\gamma_1, \tau_1], [\gamma_2, \tau_2] \in V$, we define

$$([\gamma_1, \tau_1], [\gamma_2, \tau_2])_V = (2\mu + \lambda^\star)(\gamma_1, \gamma_2)_{L_2(0,L)} + \frac{5}{6}\mu(\tau_1, \tau_2)_{L_2(0,L)}.$$

We define the operators by

$$A(\theta, u, w) = \rho(\theta, u, w) \quad \forall\, (\theta, u, w) \in H,$$

which is just the flexural strain operator, and

$$B(\theta, u, w) = [\gamma(u, w), \tau(\theta, u, w)] \ \ \forall \ (\theta, u, w) \in H,$$

which combines the membrane and shear strains engendered by the displacement functions.

The equivalence (2.3.5) that was established in Theorem 2.3.1 guaranteed the condition (2.5.2). To use the abstract results, we also need to show that the operator B is surjective. To this end, it is convenient to consider the dual operator B^* of B. It is easy to see that

$$B^* : [L_2(0, L]^2 \longrightarrow [H^{-1}(0, L)]^3,$$

$$B^*(\zeta, \eta) = (\eta, b\eta - \partial\zeta, -\partial\eta - b\zeta) \ \ \forall \ (\zeta, \eta) \in [L_2(0, L]^2.$$

We have

LEMMA 2.5.1. *If the curvature b of the middle curve S of the cross section of the cylindrical shell is not identically equal to zero, then the dual operator B^* is injective and has closed range.*

Proof. If $(\zeta, \eta) \in \ker B^*$, then

$$\|\eta\|_{-1} = 0, \quad \|b\eta - \partial\zeta\|_{-1} = 0, \quad \text{and} \quad \|\partial\eta + b\zeta\|_{-1} = 0,$$

so we have

$$\eta = 0, \quad \text{and} \quad \|\partial\zeta\|_{-1} = 0, \quad \|b\zeta\|_{-1} = 0.$$

Since the curvature b is not identically equal to zero, we must have $\zeta = 0$.

By viewing B^* as the operator A_1 in Lemma 2.3.2, and considering the compact operator

$$A_2 : [L_2(0, L)]^2 \longrightarrow [H^{-1}(0, L)]^3$$

defined by $A_2(\eta, \zeta) = (0, b\eta, b\zeta)$, the desired result will follow from lemma 2.3.2. □

The statement that the operator B is surjective then follows from the closed range theorem.

Remark 2.5.1. *If the curvature b is identically equal to zero, the operator B is still surjective, but the range will be $[L_2(0, L)/\mathbb{R}] \times L_2(0, L)$. All the results of this section still apply.*

In accordance with the abstract theory, when the shell thickness tends to zero, the behavior of the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$ can be dramatically different for whether

$$\boldsymbol{f}_0|_{\ker B} \neq 0 \tag{2.5.9}$$

or

$$\boldsymbol{f}_0|_{\ker B} = 0. \tag{2.5.10}$$

We assume $\boldsymbol{f}_0 \neq 0$, otherwise, the model is reduced down to a problem loaded by $\epsilon^2\, \boldsymbol{f}_1$, and all the analysis can be likewisely carried out and the convergence theorem in the relative energy norm can also be proved.

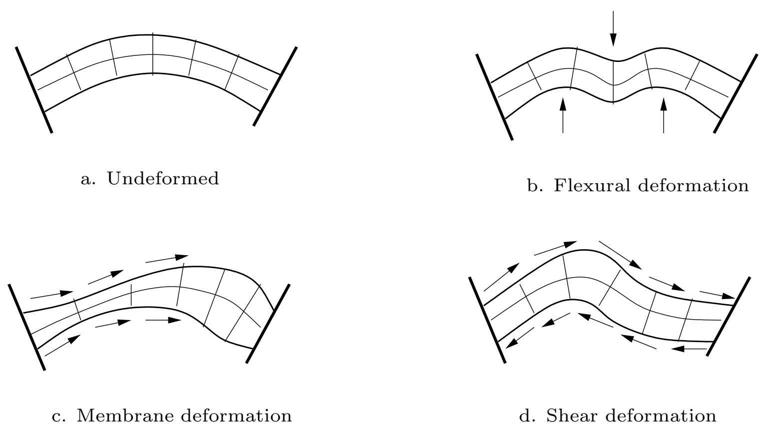


Fig. 2.2. Deformations of a cylindrical shell

Since the geometry of the middle surface of a cylindrical shell and the two sides clamping boundary condition together do not inhibit pure flexural deformation ($\ker B \neq 0$), a plane strain cylindrical shell problem can be classified as a flexural shell. However the behavior of the shell is very different depending on whether or not the applied forces make the pure flexural deformation happen. Similar situations for the second case arise in stretching a plate, or twisting a plate by tangential surface forces that are equal in magnitude but opposite in direction on the upper and lower surfaces. If the applied forces do bring about the non-inhibited asymptotically pure flexural deformation, see Figure 2.2 (b), the flexural energy will dominate membrane and shear strain energies.

If the applied force does not make the pure flexural deformation happen, as shown by Figure 2.2 (c) (d), the membrane and shear strain energies together will dominate the flexural energy. Since their magnitudes might be the same, there is no way for us to distinguish the membrane and shear energies. For this reason, and for consistency with terminologies in general shell theory, we call the first case the case of flexural shells, and the second one the membrane–shear shells.

For a flexural shell, the solution blows up at the rate of $O(\epsilon^{-2})$. To get an accurate grasp of the model solution behavior, we need to scale the loading functional as we did for the abstract problem. This scaling is equivalently imposed on the applied force functions by assuming

$$p_o^\alpha = \epsilon^2 P_o^\alpha, \quad p_e^\alpha = \epsilon^2 P_e^\alpha, \quad q_a^\alpha = \epsilon^2 Q_a^\alpha, \quad q_m^\alpha = \epsilon^2 Q_m^\alpha, \tag{2.5.11}$$

with $P_o^\alpha, P_e^\alpha, Q_a^\alpha, Q_m^\alpha$ single variable functions independent of ϵ. The resultant loading functionals are accordingly scaled as $\boldsymbol{f}_0 = \epsilon^2 \boldsymbol{F}_0$, $\boldsymbol{f}_1 = \epsilon^2 \boldsymbol{F}_1$, with $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ independent of ϵ. The expressions for $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ are the same as (2.3.3) and (2.3.4), had the lower case letters been replaced by capital letters. By the estimate (2.5.4) we have

$$\|\rho^\epsilon - \rho^0\|_{L_2(0,L)} + \epsilon^{-1} \|\gamma^\epsilon\|_{L_2(0,L)} + \epsilon^{-1} \|\tau^\epsilon\|_{L_2(0,L)} \lesssim \epsilon, \tag{2.5.12}$$

in which $\rho^0 = \rho(\theta^0, u^0, w^0)$, and $(\theta^0, u^0, w^0) \in \ker B$ is the solution of the limit problem

$$\frac{1}{3}(2\mu + \lambda^\star) \int_0^L \rho(\theta^0, u^0, w^0)\rho(\phi, y, z)dx = \langle \boldsymbol{F}_0, (\phi, y, z)\rangle \quad \forall\ (\phi, y, z) \in \ker B,$$
$$(\theta^0, u^0, w^0) \in \ker B. \tag{2.5.13}$$

This limit problem is nothing else but the limit flexural shell model. Since $\boldsymbol{F}_0|_{\ker B} \neq 0$, we have $\rho^0 \neq 0$.

For a membrane–shear shell, when $\epsilon \to 0$, the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$ converges to a finite limit. In this case, the resultant loading functional can be reformulated as

$$\langle \boldsymbol{f}_0, (\phi, y, z)\rangle = (\zeta^0, B(\phi, y, z))_V \ \ \forall\ (\phi, y, z) \in H.$$

Note that if the curvature b is not identically equal to zero, the strain operator $\gamma(u, w) = \partial u - bw$ is surjective from $[H_0^1(0, L)]^2$ to $L_2(0, L)$. Recalling the expression (2.3.3):

$$\langle \boldsymbol{f}_0, (\phi, y, z)\rangle = \frac{5}{6}\int_0^L p_o^1 \tau(\phi, y, z)dx - \frac{\lambda}{2\mu + \lambda}\int_0^L p_o^2 \gamma(y, z)$$
$$+ \int_0^L [(p_e^1 + q_a^1 - 2bp_o^1)y + (p_e^2 + q_a^2 + \partial p_o^1)z]dx,$$

we see that the condition $\boldsymbol{f}_0|_{\ker B} = 0$ is equivalent to that the bounded linear functional defined by the second term in the right hand side of this equation vanishes on the kernel of the strain operator γ. Therefore, by the closed range theorem, the condition $\boldsymbol{f}_0|_{\ker B} = 0$

is equivalent to the unique existence of $\gamma^0 \in L_2(0,L)$, such that

$$\langle \boldsymbol{f}_0, (\phi, y, z)\rangle = (2\mu + \lambda^\star) \int_0^L \gamma^0 \gamma(y,z) dx + \frac{5}{6}\mu \int_0^L \frac{1}{\mu} p_o^1 \tau(\phi, y, z) dx \quad \forall\ (\phi, y, z) \in H.$$

Recalling the definition of inner product in the space V, it is readily seen that the element $\zeta^0 \in V$ in the abstract theory takes the form

$$\zeta^0 = (\gamma^0, \frac{1}{\mu} p_o^1).$$

By the estimate (2.5.7), we get

$$\|\rho^\epsilon - \rho^0\|_{L_2(0,1)} + \epsilon^{-1} \|\gamma^\epsilon - \gamma^0\|_{L_2(0,L)} + \epsilon^{-1} \|\tau^\epsilon - \frac{1}{\mu} p_o^1\|_{L_2(0,L)} \lesssim \epsilon, \tag{2.5.14}$$

in which $\rho^0 = \rho(\theta^0, u^0, w^0)$ with $(\theta^0, u^0, w^0) \in H$ be the limit of the solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$. Actually, we also have $\gamma^0 = \gamma(\theta^0, u^0, w^0)$ and $\frac{1}{\mu} p_o^1 = \tau(\theta^0, u^0, w^0)$.

By their definitions (2.4.9), (2.4.10), and Theorem 3.3.6 of Chapter 3, we have the following estimates on the correction functions w_1 and w_2.

$$\epsilon \|\partial w_1\|_{L_2(0,L)} + \|\frac{1}{2\mu + \lambda^\star}(p_o^2 - \frac{\lambda}{2\mu + \lambda} \sigma_0^{11}) - w_1\|_{L_2(0,L)} \lesssim \epsilon^{1/2} [\|p_o^2\|_{H^1(0,L)} + \|\sigma_0^{11}\|_{H^1(0,L)}] \tag{2.5.15}$$

and

$$\epsilon \|\partial w_2\|_{L_2(0,L)} + \|\frac{1}{2\mu+\lambda^\star}(p_e^2 - \frac{\lambda}{2\mu+\lambda}\sigma_1^{11}) - 2w_2\|_{L_2(0,L)}$$
$$\lesssim \epsilon^{1/2}[\|p_e^2\|_{H^1(0,L)} + \|\sigma_1^{11}\|_{H^1(0,L)}]. \quad (2.5.16)$$

2.5.4 Convergence theorem

With all the above preparations, we are ready to prove the convergence theorem. We denote the energy norm of a stress field $\underset{\approx}{\sigma}$ and a strain field $\underset{\approx}{\chi}$ defined on the shell cross section Ω^ϵ by

$$\|\underset{\approx}{\sigma}\|_{E^\epsilon} = (\int_{\Omega^\epsilon} A_{\alpha\beta\lambda\gamma}\sigma^{\lambda\gamma}\sigma^{\alpha\beta})^{1/2} \quad \text{and} \quad \|\underset{\approx}{\chi}\|_{E^\epsilon} = (\int_{\Omega^\epsilon} C^{\alpha\beta\lambda\gamma}\chi_{\lambda\gamma}\chi_{\alpha\beta})^{1/2},$$

respectively. Since the elasticity tensor $C^{\alpha\beta\lambda\gamma}$ and the compliance tensor $A_{\alpha\beta\lambda\gamma}$ are uniformly positive definite and bounded, the energy norms are equivalent to the sums of the $L_2(\omega^\epsilon)$ norms of the tensor components.

THEOREM 2.5.2. *Assume that the surface force functions have the regularity $\tilde{p}_\pm^\alpha \in H^1(0,L)$ and the body force functions q_a^α, $q_m^\alpha \in L_2(0,L)$. Let $\underset{\approx}{\sigma}^*$ be the actual stress distribution over the loaded shell, and $\underset{\sim}{v}^*$ the true displacement field arising in response to the applied forces and boundary conditions. Based on the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$, we define the statically admissible stress field $\underset{\approx}{\sigma}$ by the formulae (2.4.7), and define the kinematically admissible displacement field $\underset{\sim}{v}$ by the formulae (2.4.11). We have the* *estimate*

$$\frac{\|\underset{\approx}{\sigma}^* - \underset{\approx}{\sigma}\|_{E^\epsilon} + \|\underset{\approx}{\chi}(\underset{\sim}{v}^*) - \underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}}{\|\underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}} \lesssim \epsilon^{1/2}\,. \tag{2.5.17}$$

Proof. The proof is based on the two energies principle, the formulae for the constitutive residual (2.4.14) – (2.4.16), the asymptotic behaviors (2.5.12) and (2.5.14) of the model solution, and the estimates (2.5.15) and (2.5.16) on the correction functions. Since the behaviors of the model solution are very different for flexural shells and membrane–shear shells, we prove the theorem for the two cases separately. In the following, we will simply denote the norm $\|\cdot\|_{L_2(\omega^\epsilon)}$ by $\|\cdot\|$.

Flexural shells

This is the case in which the solution blows up at the rate of $O(\epsilon^{-2})$. To ease the analysis, we scale the loading functions by assuming that (2.5.11) holds, with $P_o^\alpha, P_e^\alpha, Q_a^\alpha, Q_m^\alpha$ single variable functions independent of ϵ. Note that, since we are considering the relative error estimate, this scaling is not a real restriction on the applied force functions. With this scaling, we have the estimate (2.5.12), from which, we get

$$\|\rho^\epsilon - \rho^0\|_{L_2(0,L)} \lesssim \epsilon, \quad \|\gamma^\epsilon\|_{L_2(0,L)} \lesssim \epsilon^2, \quad \|\tau^\epsilon\|_{L_2(0,L)} \lesssim \epsilon^2\,. \tag{2.5.18}$$

From the equivalence (2.3.5), we get

$$\|\theta^\epsilon\|_{H^1(0,L)} + \|u^\epsilon\|_{H^1(0,L)} + \|w^\epsilon\|_{H^1(0,L)} \simeq 1 \simeq \|\rho^0\|_{L_2(0,L)}. \tag{2.5.19}$$

By the definition (2.4.1), we have

$$\begin{aligned}\sigma_0^{11} &= \frac{1}{3}b\,\epsilon^2\,\sigma_1^{11} + (2\mu+\lambda^\star)\gamma^\epsilon + \frac{\lambda}{2\mu+\lambda}\,\epsilon^2\,P_o^2,\\ \sigma_1^{11} &= (2\mu+\lambda^\star)\rho^\epsilon + \frac{\lambda}{2\mu+\lambda}\,\epsilon^2(P_e^2+bP_o^2),\\ \sigma_0^{12} &= \frac{5}{4}\mu\tau^\epsilon - \frac{5}{4}\epsilon^2\,P_o^1 + \frac{1}{4}b\epsilon^4 P_e^1,\end{aligned}\tag{2.5.20}$$

from which, we have the estimates

$$\|\sigma_0^{11}\|_{L_2(0,L)} \lesssim \epsilon^2, \quad \|\sigma_1^{11}\|_{L_2(0,L)} \simeq 1, \quad \|\sigma_0^{12}\|_{L_2(0,L)} \lesssim \epsilon^2\,. \tag{2.5.21}$$

By the estimate (2.4.4), we have

$$\|\sigma_2^{11}\|_{H^1(0,L)} \lesssim \epsilon^2\,. \tag{2.5.22}$$

From the first and last equations of (2.4.2), we see the estimates

$$\|\sigma_0^{11}\|_{H^1(0,L)} \lesssim \epsilon^2, \quad \|\sigma_1^{11}\|_{H^1(0,L)} \simeq 1. \tag{2.5.23}$$

From the last two equations in (2.4.3), we see the estimates

$$\|\sigma_0^{22}\|_{L_2(0,L)} \lesssim \epsilon^2, \quad \|\sigma_1^{22}\|_{L_2(0,L)} \lesssim \epsilon^3\,. \tag{2.5.24}$$

By the estimates on the correction functions (2.5.15) and (2.5.16), we have

$$\begin{aligned} &\epsilon\,\|\partial w_1\|_{L_2(0,L)} + \|\frac{1}{2\mu+\lambda^\star}(\epsilon^2\,P_o^2 - \frac{\lambda}{2\mu+\lambda}\sigma_0^{11}) - w_1\|_{L_2(0,L)} \lesssim \epsilon^{5/2}, \\ &\epsilon\,\|\partial w_2\|_{L_2(0,L)} + \|\frac{1}{2\mu+\lambda^\star}(\epsilon^2\,P_e^2 - \frac{\lambda}{2\mu+\lambda}\sigma_1^{11}) - 2w_2\|_{L_2(0,L)} \lesssim \epsilon^{1/2}\,. \end{aligned} \tag{2.5.25}$$

From the equation (2.4.12), we see that in the expression of $\chi_{11}(\underset{\sim}{v})$, the term $t\rho(\theta^\epsilon, u^\epsilon, w^\epsilon)$ dominates in $L_2(\omega^\epsilon)$, and by (2.5.18), we get the lower bound $\|\chi_{11}(\underset{\sim}{v})\|^2 \gtrsim \epsilon^3$, and so

$$\|\underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}^2 \gtrsim \epsilon^3\,. \tag{2.5.26}$$

By the two energies principle, we have

$$\begin{aligned} \|\underset{\approx}{\sigma}^* - \underset{\approx}{\sigma}\|_{E^\epsilon}^2 + \|\underset{\approx}{\chi}(\underset{\sim}{v}^*) - \underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}^2 &= \int_{\Omega^\epsilon} C^{\alpha\beta\lambda\gamma}\varrho_{\lambda\gamma}\varrho_{\alpha\beta} \\ &\lesssim \|\varrho_{11}\|^2 + \|\varrho_{12}\|^2 + \|\varrho_{22}\|^2. \end{aligned} \tag{2.5.27}$$

In the expression (2.4.14) of ϱ_{11}, we can see that the square integrals over ω^ϵ of all the terms are bounded by $O(\epsilon^5)$, therefore, we have $\|\varrho_{11}\|^2 \lesssim \epsilon^5$. From the expression (2.4.15) of ϱ_{12}, we see that the square integrals of all the terms are bounded by $O(\epsilon^4)$, and so we have $\|\varrho_{12}\|^2 \lesssim \epsilon^4$. From the expression (2.4.16) of ϱ_{22}, we see the bounds are $O(\epsilon^4)$, and so $\|\varrho_{22}\|^2 \lesssim \epsilon^4$.

Therefore, by (2.5.27), we get the upper bound

$$\|\underset{\approx}{\sigma}^* - \underset{\approx}{\sigma}\|_{E^\epsilon}^2 + \|\underset{\approx}{\chi}(\underset{\sim}{v}^*) - \underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}^2 \lesssim \epsilon^4 \tag{2.5.28}$$

The conclusion of the theorem for the case of flexural shells then follows from the lower bound (2.5.26) and this upper bound.

Membrane–shear shells

In this case, under the assumption that p_o^α, p_e^α, q_a^α, and q_m^α are independent of ϵ, the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$ converges to a finite limit in the space H when $\epsilon \to 0$, so we have

$$\|\theta^\epsilon\|_{H^1(0,L)} + \|u^\epsilon\|_{H^1(0,L)} + \|w^\epsilon\|_{H^1(0,L)} \lesssim 1.$$

From the estimate (2.5.14), we get

$$\|\rho^\epsilon\|_{L_2(0,L)} \lesssim 1, \quad \|\gamma^\epsilon - \gamma^0\|_{L_2(0,L)} \lesssim \epsilon^2, \quad \|\mu\tau^\epsilon - p_o^1\|_{L_2(0,L)} \lesssim \epsilon^2. \tag{2.5.29}$$

Since $\zeta^0 \neq 0$, we know that γ^0 and p_o^1 can not be zero simultaneously. From the equation (2.4.1) we see

$$\|\sigma_0^{11}\|_{L_2(0,L)} \lesssim 1, \quad \|\sigma_1^{11}\|_{L_2(0,L)} \lesssim 1, \quad \|\sigma_0^{12}\|_{L_2(0,L)} \lesssim \epsilon^2. \tag{2.5.30}$$

By the estimate (2.4.4), we have

$$\|\sigma_2^{11}\|_{H^1(0,L)} \lesssim \epsilon^2. \tag{2.5.31}$$

From the first and last two equations of (2.4.2), we see the estimates

$$\|\sigma_0^{11}\|_{H^1(0,L)} \lesssim 1, \quad \|\sigma_1^{11}\|_{H^1(0,L)} \lesssim 1. \tag{2.5.32}$$

From the last two equations in (2.4.3), we see the estimates

$$\|\sigma_0^{22}\|_{L_2(0,L)} \lesssim \epsilon^2, \quad \|\sigma_1^{22}\|_{L_2(0,L)} \lesssim \epsilon\,. \tag{2.5.33}$$

By the estimates on the correction functions (2.5.15) and (2.5.16), we have

$$\begin{aligned} &\epsilon\,\|\partial w_1\|_{L_2(0,L)} + \|\frac{1}{2\mu+\lambda^\star}(p_o^2 - \frac{\lambda}{2\mu+\lambda}\sigma_0^{11}) - w_1\|_{L_2(0,L)} \lesssim \epsilon^{1/2}, \\ &\epsilon\,\|\partial w_2\|_{L_2(0,L)} + \|\frac{1}{2\mu+\lambda^\star}(p_e^2 - \frac{\lambda}{2\mu+\lambda}\sigma_1^{11}) - 2w_2\|_{L_2(0,L)} \lesssim \epsilon^{1/2}\,. \end{aligned} \tag{2.5.34}$$

From the equation (2.4.12), we see that in the expression of $\chi_{11}(\underset{\sim}{v})$, the term $\gamma(u^\epsilon, w^\epsilon)$ dominates, and in the expression of $\chi_{12}(\underset{\sim}{v})$, the term $\frac{1}{2}\tau(\theta^\epsilon, u^\epsilon, w^\epsilon)$ dominates. Asymptotically, we have the equivalency

$$\|\gamma^\epsilon\|_{L_2(0,L)} + \|\tau^\epsilon\|_{L_2(0,L)} \simeq \|\gamma^0\|_{L_2(0,L)} + \|\frac{1}{\mu}p_o^1\|_{L_2(0,L)} \simeq 1. \tag{2.5.35}$$

We get the lower bound $\|\chi_{11}(\underset{\sim}{v})\|^2 + \|\chi_{12}(\underset{\sim}{v})\|^2 \gtrsim \epsilon$, and so

$$\|\underset{\approx}{\chi}(\underset{\sim}{v})\|^2_{E^\epsilon} \gtrsim \epsilon\,. \tag{2.5.36}$$

In the expression (2.4.14) of ϱ_{11}, we can see that the square integrals over ω^ϵ of all the terms are bounded by $O(\epsilon^3)$, therefore, we have $\|\varrho_{11}\|^2 \lesssim \epsilon^3$. From the expression (2.4.15) of ϱ_{12}, we see all the terms are bounded by $O(\epsilon^2)$, and so we have $\|\varrho_{12}\|^2 \lesssim \epsilon^2$. From the expression (2.4.16) of ϱ_{22}, we see the bound is $\|\varrho_{22}\|^2 \lesssim \epsilon^2$.

By the two energies principle, we have

$$\|\underset{\approx}{\sigma}^* - \underset{\approx}{\sigma}\|_{E^\epsilon}^2 + \|\underset{\approx}{\chi}(\underset{\sim}{v}^*) - \underset{\approx}{\chi}(\underset{\sim}{v})\|_{E^\epsilon}^2 \lesssim \|\varrho_{11}\|^2 + \|\varrho_{12}\|^2 + \|\varrho_{22}\|^2 \lesssim \epsilon^2 . \qquad (2.5.37)$$

The conclusion of the theorem for the case of membrane–shear shells then follows from the lower bound (2.5.36) and this upper bound. □

2.6 Shear dominated shell examples

To emphasize the necessity of using the Naghdi-type model in some cases, we give two examples for which the model equations are explicitly solvable. For these problems, the Koiter model and limiting flexural model only give solutions that are identically equal to zero, while our Naghdi-type model can very well capture the shear dominated deformations.

2.6.1 A beam problem

a. Undeformed b. Shear dominated deformation

Fig. 2.3. Shear dominated deformation of a beam

We consider a special plane strain cylindrical shell whose cross section is a thin rectangle with thickness 2ϵ and length $L = 1$. The curvature of the middle curve then is $b \equiv 0$. The applied forces are: $\underset{\sim}{q} = 0$, $\underset{\sim}{p}_{\pm} = \pm \boldsymbol{a}_1$. The leading term of the resultant loading functional (2.3.3) is given by

$$\langle \boldsymbol{f}_0, (\phi, y, z)\rangle = \frac{5}{6}\int_0^1 \tau(\phi, y, z)dx.$$

The condition $\boldsymbol{f}_0|_{\ker B} = 0$ is obviously satisfied. Therefore the limiting flexural shell model only gives a zero solution. So does Koiter's model.

The model (2.3.2), written in differential form, reduces to

$$-\frac{1}{3}\epsilon^2(2\mu+\lambda^\star)\partial^2\theta^\epsilon + \frac{5}{6}\mu(\theta^\epsilon + \partial w^\epsilon) = \frac{5}{6},$$

$$-(2\mu+\lambda^\star)\partial^2 u^\epsilon = 0, \quad -\frac{5}{6}\mu\partial(\theta^\epsilon + \partial w^\epsilon) = 0, \tag{2.6.1}$$

$$(\theta^\epsilon, u^\epsilon, w^\epsilon) \in [H_0^1(0,1)]^3,$$

which is just the Timoshenko beam bending and stretching model [3]. The solution is given by

$$\theta^\epsilon = c^\epsilon x(1-x), \quad u^\epsilon = 0, \quad w^\epsilon = -\frac{1}{3}c^\epsilon(x-\frac{1}{2})x(1-x),$$

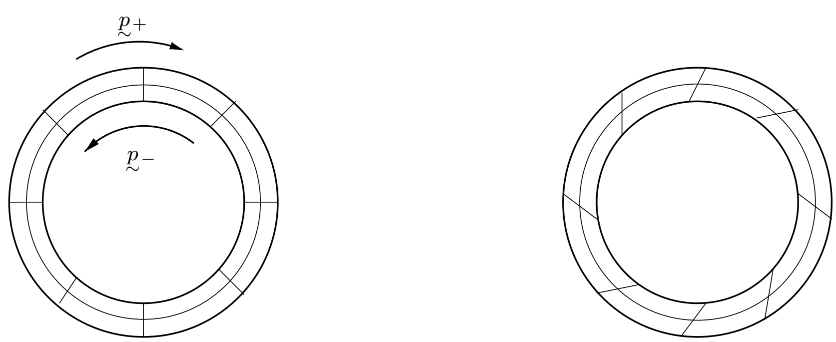
where $c^\epsilon = [\frac{\mu}{6} + \frac{16\,\epsilon^2\,\mu(\mu+\lambda)}{5(2\mu+\lambda)}]^{-1}$. We see the convergences:

$$\lim_{\epsilon\to 0}\theta^\epsilon = \frac{1}{\mu}6x(1-x), \quad \lim_{\epsilon\to 0} w^\epsilon = -2\frac{1}{\mu}(x-\frac{1}{2})x(1-x), \quad \lim_{\epsilon\to 0}(\theta^\epsilon + \partial w^\epsilon) = \frac{1}{\mu}.$$

This is basically the asymptotic pattern of the exact deformation of the elastic body. Note that the last convergence shows that the transverse shear strain tends to a finite limit, a violation of the Kirchhoff–Love hypothesis.

2.6.2 A circular cylindrical shell problem

In this subsection, we consider a plane strain circular cylindrical shell problem. The shell occupies an infinitely long circular cylinder whose thickness is $2\,\epsilon$. The middle curve of the cross section Ω^ϵ is the unit circle whose curvature is $b = -1$. The shell is loaded by surface forces whose densities are $\underset{\sim}{p}_{\pm} = \pm(1 \mp \epsilon)^2 \boldsymbol{a}_1$, and a body force whose contravariant components are given by $q^1 = \dfrac{12\,\epsilon^2}{(1+t)^2} r(t), \quad q^2 = 0.$



a. Undeformed　　　　b. Deformed

Fig. 2.4. Shear dominated deformation of a circular cylinder

It can be verified that both the net force and net torque resulting from the applied forces are zero. Therefore the surface and body forces are compatible and the problem is well defined. By using the formulae (2.2.21) to compute q_a^α and q_m^α, the resultant loading functional in the model can be computed. We have

$$\langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\phi, y, z)\rangle = \frac{5}{6}\int_0^{2\pi} \tau(\phi, y, z) dx$$
$$+ \epsilon^2 [\frac{1}{2}\int_0^{2\pi} \tau(\phi, y, z) dx + 2\int_0^{2\pi} y dx - 2\int_0^{2\pi} \phi dx] + \epsilon^4 \langle \boldsymbol{r}, (\phi, y, z)\rangle,$$

where $\boldsymbol{r}$ is a functional independent of ϵ. The higher order term $O(\epsilon^4 \boldsymbol{r})$ is provably negligible. With this higher order term cutoff, we will have $\boldsymbol{f}_0|_{\ker B} = 0$. The model solution then is

$$u^\epsilon = 0, \quad w^\epsilon = 0, \quad \theta^\epsilon = \frac{1}{\mu} - \frac{9}{5\mu}\epsilon^2,$$

which gives a displacement field that is purely rotational. The covariant components of the displacement field provided by this model are $v_1 = (1+t)t\theta^\epsilon, \quad v_2 = 0$. The covariant components of the strain tensor engendered by this displacement field are, by the formulae (2.2.26),

$$\chi_{11} = 0, \quad \chi_{22} = 0, \quad \chi_{12} = \frac{1}{2}\theta^\epsilon = \frac{1}{2\mu} - \frac{9}{10\mu}\epsilon^2 . \tag{2.6.2}$$

It can be easily checked that the stress field whose contravariant components are given by

$$\sigma^{11} = 0, \quad \sigma^{22} = 0, \quad \sigma^{12} = \frac{1}{1+t}[1 - 2t + 3t^2 - 2\epsilon^2] \tag{2.6.3}$$

is statically admissible. By using the formulae (2.2.19), we see that for the admissible stress field defined by (2.6.3),

$$A_{11\lambda\gamma}\sigma^{\lambda\gamma} = A_{22\lambda\gamma}\sigma^{\lambda\gamma} = 0, \quad A_{12\lambda\gamma}\sigma^{\lambda\gamma} = \frac{1}{2\mu}(1+t)[1-2t+3t^2-2\,\epsilon^2].$$

Therefore, the the constitutive residual can be bounded by

$$\varrho_{11} = 0, \quad \varrho_{22} = 0, \quad |\varrho_{12}| \lesssim \epsilon\,.$$

From the two energies principle, we know that the pure rotational displacement given by the model is very close to the exact displacement of this circular cylinder arising in response to the applied forces. The error in the relative energy norm is $O(\epsilon)$. The shear strain and stress absolutely dominate all the other strain and stress components. For this problem, the Koiter model and the limiting flexural shell model only give a solution that is identically equal to zero. This is a case for which the Naghdi-type model is indispensable.

Chapter 3

Analysis of the parameter dependent variational problems

3.1 Introduction

The plane strain cylindrical shell model (2.3.2) that we justified in the last chapter can be put in the form of the abstract ϵ-dependent variational problem (3.2.2) below, and Theorem 3.3.1 was essential to the justification. This abstract problem also applies to the spherical shell model and general shell model that we are going to derive and justify. It is the purpose of this chapter to establish all the *a priori* estimates that are necessary for our analyses. The behavior of the solution of such ϵ-dependent can be drastically different in different circumstances. We will classify the problem on the abstract level at the end of this chapter. Results that will be used to analyze the relations between our model and other existing shell theories will also be given.

3.2 The parameter dependent problem and its mixed formulation

For a Hilbert space X, we denote its dual by X^*, and for any $f \in X^*$, we use $i_X f \in X$ to denote its Riesz representation. The isomorphism $\pi_X : X \to X^*$ is defined as the inverse of i_X, and is equal to i_{X^*} under the usual identification of X and X^{**}.

Let H, U, and V be Hilbert spaces, A and B be linear continuous operators from H to U and V, respectively. We assume

$$\|Au\|_U^2 + \|Bu\|_V^2 \simeq \|u\|_H^2 \quad \forall\, u \in H. \tag{3.2.1}$$

By properly defining spaces and operators, the shell models we derive can be written in the form of the ϵ-dependent variational problem:

$$\begin{aligned} &\epsilon^2 (Au, Av)_U + (Bu, Bv)_V = \langle f_0 + \epsilon^2 f_1, v \rangle, \\ &u \in H, \quad \forall\, v \in H, \end{aligned} \tag{3.2.2}$$

where $f_0, f_1 \in H^*$ are two functionals independent of ϵ, and $f_0 \neq 0$. It turns out that, in all the cases we are going to analyze, when $\epsilon \to 0$, the asymptotic behavior of solution of this variational problem is mostly determined by the leading term f_0. For this reason, we first analyze the abstract problem

$$\begin{aligned} &\epsilon^2 (Au, Av)_U + (Bu, Bv)_V = \langle f, v \rangle, \\ &u \in H, \quad \forall\, v \in H, \end{aligned} \tag{3.2.3}$$

with $f \in H^*$ independent of ϵ. The behavior of the solution of (3.2.2) will be obtained by a simple argument once the simpler problem (3.2.3) is fully understood.

The variational problem (3.2.3) represents the Timoshenko beam bending model and Reissner–Mindlin plate bending model, with u standing for the transverse deflection of the middle surface and rotation of normal fibers, and Au the bending strain and Bu

the transverse shear strain engendered by u. The Koiter shell model, which adopts the Kirchhoff–Love assumption and so ignores the transverse shear deformation, takes this form, with the variable u representing the middle surface displacement, Au the flexural strain, and Bu the membrane strain. The Naghdi shell model, and the variant we derive, can be put in this form if we let u be the middle surface displacement and normal fiber rotation. The operator A defines the flexural strain and B combines the transverse shear and membrane strains engendered by u. The spaces H is a multiple L_2-based first order Sobolev space, and U and V are equivalent to L_2 or products of L_2. Referring to the physical background of the abstract problem, we will call $\epsilon^2(Au, Au)_U$ the flexural energy, and $(Bu, Bu)_V$ the membrane–shear energy engendered by the displacement function u.

Under the assumption (3.2.1), for any $f \in H^*$, the variational problem (3.2.3) has a unique solution depending on ϵ. In what follows, whenever the ϵ dependence needs to be emphasized, the solution will be denoted by u^ϵ. We are concerned with the behavior of the solution of such problems, especially when ϵ is small.

If we set $F = \epsilon^{-2} f$, the following rough estimate is obvious

$$\|u^\epsilon\|_H \lesssim \|F\|_{H^*}. \tag{3.2.4}$$

We will derive more accurate estimates on the solution of (3.2.3) by introducing a mixed formulation. In what follows, we will need some basic results.

First we recall that if X and Y are Hilbert spaces with $X \subset Y$, and if X is dense in Y, then the restriction operator defines an injection of Y^* onto a dense subspace of X^* (and we usually identify Y^* with that dense subspace).

We next recall the sum and intersection constructions for Hilbert spaces. If Hilbert spaces X and Y are both continuously included in a larger Hilbert space, then the intersection $X \cap Y$ and the sum $X + Y$ are themselves Hilbert spaces with the norms

$$\|z\|_{X \cap Y} = (\|z\|_X^2 + \|z\|_Y^2)^{1/2} \quad \text{and} \quad \|z\|_{X+Y} = \inf_{z=x+y} (\|x\|_X^2 + \|y\|_Y^2)^{1/2},$$

and we have

Lemma 3.2.1. *If in addition, $X \cap Y$ is dense in both X and Y, then the dual spaces X^* and Y^* can be viewed as subspace of $(X \cap Y)^*$ and we have*

$$X^* + Y^* = (X \cap Y)^*.$$

The operator $B : H \to V$ may have closed range in some problems, as in the cases of Timoshenko beam bending model, the plain strain cylindrical shell model of Chapter 2 and other 1D models. This operator may have a range that is not closed in V, as in the Reissner–Mindlin plate, Koiter and Naghdi shell models, as well as numerous singular perturbation problems.

Let $W = B(H) \subset V$ be the range of B, whose norm is defined by, for any $\zeta \in W$,

$$\|\zeta\|_W = \inf_{\zeta = Bu} \|u\|_H. \tag{3.2.5}$$

With this norm, W is a Hilbert space isomorphic to $H/\ker B$. This space plays a crucial role in the following analysis. For the Reissner–Mindlin bending model of a totally clamped plate, this space is $\overset{\circ}{\underset{\sim}{H}}(\mathrm{rot})$. Without loss of generality we may assume that W is dense in V, otherwise, we can just replace V by the closure of W in it.

Associated with a Hilbert space X and any positive number ς, we define the Hilbert spaces ςX. As set, ςX equals to X, but the norm is defined by $\|x\|_{\varsigma X} = \varsigma\|x\|_X$.

Since W is dense in V, so V^* and ϵV^* are dense in W^*. The dual space of $W^* \cap \epsilon V^*$ is, by Lemma 3.2.1, $W + \epsilon^{-1} V$.

If u^ϵ solves (3.2.3) and $\xi^\epsilon = \epsilon^{-2} \pi_V B u^\epsilon \in V^*$, then $(u^\epsilon, \xi^\epsilon)$ solves the mixed problem

$$
\begin{aligned}
&(Au, Av)_U + \langle \xi, Bv \rangle = \langle F, v \rangle \quad \forall\, v \in H, \\
&\langle \eta, Bu \rangle - \epsilon^2 (\xi, \eta)_{V^*} = 0 \quad \forall\, \eta \in V^*, \\
&u \in H, \quad \xi \in V^*.
\end{aligned}
\tag{3.2.6}
$$

For this mixed problem, we have the following result

Theorem 3.2.2. *The mixed problem (3.2.6) has a unique solution $(u^\epsilon, \xi^\epsilon) \in H \times V^*$, and the equivalence*

$$
\|u^\epsilon\|_H + \|\xi^\epsilon\|_{W^* \cap \epsilon V^*} \simeq \|F\|_{H^*} \tag{3.2.7}
$$

holds.

Proof. The pair $(u^\epsilon, \xi^\epsilon)$ solves (3.2.6) if and only if $\xi^\epsilon = \epsilon^{-2} \pi_V B u^\epsilon$ and u^ϵ solves (3.2.3), so the existence and uniqueness are established. From (3.2.4) and the first equation, we get

$$\|u^\epsilon\|_H + \|\xi^\epsilon\|_{W^*} \lesssim \|F\|_{H^*}.$$

Taking $v = u^\epsilon$ in the first equation and $\eta = \xi^\epsilon$ in the second equation, we get the bound on $\|\xi^\epsilon\|_{\epsilon V^*}$. From the first equation, we easily get that $\|F\|_{H^*} \lesssim \|u^\epsilon\|_H + \|\xi^\epsilon\|_{W^*}$. □

This theorem shows that the right space for the auxiliary variable ξ^ϵ is $W^* \cap \epsilon V^*$.

To analyze the asymptotic behavior of the solution u^ϵ, we also need to consider the following general mixed problem.

$$\begin{aligned} &(Au, Av)_U + \langle \xi, Bv \rangle = \langle F, v \rangle \quad \forall\, v \in H, \\ &\langle \eta, Bu \rangle - \epsilon^2 (\xi, \eta)_{V^*} = \langle I, \eta \rangle \quad \forall\, \eta \in V^*, \\ &u \in H, \quad \xi \in V^*. \end{aligned} \tag{3.2.8}$$

here, $I \in V$. For this general mixed problem we have

Theorem 3.2.3. *The mixed problem (3.2.8) has a unique solution $(u^\epsilon, \xi^\epsilon) \in H \times V^*$, and the equivalence*

$$\|u^\epsilon\|_H + \|\xi^\epsilon\|_{W^* \cap \epsilon V^*} \simeq \|F\|_{H^*} + \|I\|_{W + \epsilon^{-1} V} \tag{3.2.9}$$

holds.

Proof. Let $\zeta_*^0 = \pi_V I$, then $\langle I, \eta\rangle = (\zeta_*^0, \eta)_{V^*}$. The problem (3.2.8) can be reformulated as

$$(Au, Av)_U + \langle \xi + \epsilon^{-2}\,\zeta_*^0, Bv\rangle = \langle F, v\rangle + \epsilon^{-2}\langle \zeta_*^0, Bv\rangle \quad \forall\, v \in H,$$

$$\langle \eta, Bu\rangle - \epsilon^2(\xi + \epsilon^{-2}\,\zeta_*^0, \eta)_{V^*} = 0 \quad \forall\, \eta \in V^*, \tag{3.2.10}$$

$$u \in H, \quad \xi \in V^*.$$

This formulation is in the form of (3.2.6), therefore, we get the existence and uniqueness of the solution from Theorem 3.2.2. In the following, C_1–C_5 are constants independent of ϵ.

From the first equation of (3.2.8), we get

$$\|\xi^\epsilon\|_{W^*} \le C_1(\|F\|_{H^*} + \|Au^\epsilon\|_U). \tag{3.2.11}$$

Taking $v = u^\epsilon$ and $\eta = \xi^\epsilon$ in (3.2.8), and subtracting the second equation from the first equation, we get

$$\|Au^\epsilon\|_U^2 + \epsilon^2\,\|\xi^\epsilon\|_{V^*}^2 = \langle F, u^\epsilon\rangle - \langle I, \xi^\epsilon\rangle,$$

so we have

$$\|Au^\epsilon\|_U^2 + \epsilon^2\,\|\xi^\epsilon\|_{V^*}^2 \le C_2(\|F\|_{H^*}\|u^\epsilon\|_H + \|I\|_{W+\epsilon^{-1}V}\|\xi^\epsilon\|_{W^*\cap\epsilon V^*}). \tag{3.2.12}$$

From the second equation of (3.2.8) and the bound $\|I\|_V \lesssim \|I\|_{W+\epsilon^{-1}V}$, we get

$$\|Bu^\epsilon\|_V \le C_3(\epsilon^2\,\|\xi^\epsilon\|_{V^*} + \|I\|_{W+\epsilon^{-1}V}). \tag{3.2.13}$$

Combining (3.2.11), (3.2.12) and (3.2.13), we get

$$\|Au^\epsilon\|_U^2 + \|Bu^\epsilon\|_V^2 + \epsilon^2 \|\xi^\epsilon\|_{V^*}^2 \le C_4(\|F\|_{H^*}\|u^\epsilon\|_H + \|I\|_{W+\epsilon^{-1}V}\|Au^\epsilon\|_U$$
$$+\|I\|_{W+\epsilon^{-1}V}\|\xi^\epsilon\|_{\epsilon V^*} + \|I\|_{W+\epsilon^{-1}V}\|F\|_{H^*} + \|I\|_{W+\epsilon^{-1}V}^2). \tag{3.2.14}$$

By using Cauchy's inequality and (3.2.1), we get

$$\|Au^\epsilon\|_U^2 + \|Bu^\epsilon\|_V^2 + \epsilon^2 \|\xi^\epsilon\|_{V^*}^2 \le C_5(\|F\|_{H^*}^2 + \|I\|_{W+\epsilon^{-1}V}^2). \tag{3.2.15}$$

The upper bound of the left hand side in (3.2.9) follows from (3.2.11) and (3.2.15). The other direction follows from the formulation (3.2.8) directly. □

This result is an extension of an equivalence theorem established for the Reissner–Mindlin plate bending model in [8].

3.3 Asymptotic behavior of the solution

When $\epsilon \to 0$, the behavior of the solution u^ϵ of (3.2.3) is dramatically different depending on whether

$$f|_{\ker B} \neq 0, \tag{3.3.1}$$

or

$$f|_{\ker B} = 0. \tag{3.3.2}$$

The asymptotic behavior needs to be discussed separately for these two cases. In the first case, the solution blows up at the rate of $O(\epsilon^{-2})$. To fix the situation, we scale the problem by assuming that $F = \epsilon^{-2} f$ is independent of ϵ. With this scaling, the problem (3.2.3) or equivalently (3.2.6) can be viewed as a penalization of the constrained problem

$$\min_{u \in \ker B} \frac{1}{2}(Au, Au)_U - \langle F, u \rangle. \tag{3.3.3}$$

This constrained problem has a unique nonzero solution $u^0 \in \ker B$. This minimization problem can also be written in mixed form as

$$\begin{aligned} &(Au, Av)_U + \langle \xi, Bv \rangle = \langle F, v \rangle \quad \forall\, v \in H, \\ &\langle \eta, Bu \rangle = 0 \quad \forall\, \eta \in W^*, \quad u \in H, \quad \xi \in W^*. \end{aligned} \tag{3.3.4}$$

This mixed problem has a unique solution (u^0, ξ^0) with $u^0 \in \ker B$ and $\xi^0 \in W^*$. And we have the equivalence

$$\|u^0\|_H + \|\xi^0\|_{W^*} \simeq \|F\|_{H^*}. \tag{3.3.5}$$

In the second case, the problem is essentially a singular perturbation problem. We do not need to scale the problem. From the definition of W, we know that B is surjective from H to W. By the closed range theorem in functional analysis, there exists a unique $\zeta_*^0 \in W^*$ such that

$$\langle f, v \rangle = \langle \zeta_*^0, Bv \rangle \quad \forall\, v \in H. \tag{3.3.6}$$

3.3.1 The case of surjective membrane–shear operator

We first discuss the simpler case in which the operator $B : H \to V$ is surjective. An example of this case is the plane strain cylindrical shell problems discussed in the last chapter. In this case, we have $W = V$, $W^* = V^*$, and

THEOREM 3.3.1. *Let H, U and V be Hilbert spaces, the linear operators $A : H \to U$ bounded, and $B : H \to V$ bounded and surjective. We assume the equivalence (3.2.1) holds, so the variational problem (3.2.3) has a unique solution $u^\epsilon \in H$.*

If $f|_{\ker B} \neq 0$, we assume $F = \epsilon^{-2} f$ is independent of ϵ. Then

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V \lesssim \epsilon \|\xi^0\|_{V^*} \lesssim \epsilon \|F\|_{H^*}, \tag{3.3.7}$$

where (u^0, ξ^0) is the solution of the ϵ-independent problem (3.3.4), with $u^0 \in \ker B$, $\xi^0 \in V^*$, and $u^0 \neq 0$.

If $f|_{\ker B} = 0$, there exists a nonzero element $\zeta_*^0 \in V^*$, such that

$$\langle f, v\rangle = \langle \zeta_*^0, Bv\rangle = (\zeta^0, Bv)_V \quad \forall\ v \in H.$$

Here $\zeta^0 \in V$ is the Riesz representation of ζ_*^0. There exists a unique $u^0 \in H$, satisfying $Bu^0 = \zeta^0$ together with the estimate

$$\|u^\epsilon - u^0\|_H \lesssim \epsilon^2 \|\zeta^0\|_V. \tag{3.3.8}$$

Moreover

$$\epsilon^{-1}\,\|Au^{\epsilon}-Au^{0}\|_{U}+\epsilon^{-1}\,\|Bu^{\epsilon}-\zeta^{0}\|_{V}\lesssim\epsilon\,\|\zeta^{0}\|_{V}=\epsilon\,\|f\|_{H^{*}}. \tag{3.3.9}$$

Proof. We prove (3.3.7) first. Under the assumption of $W = V$, the solutions of the mixed problems (3.2.6) and (3.3.4) satisfy

$$(Au^{\epsilon},Av)_{U}+\langle\xi^{\epsilon},Bv\rangle=\langle F,v\rangle\ \ \forall\ v\in H,$$

$$\langle Bu^{\epsilon},\eta\rangle-\epsilon^{2}(\xi^{\epsilon},\eta)_{V^{*}}=0\ \ \forall\ \eta\in V^{*}$$

and

$$(Au^{0},Av)_{U}+\langle\xi^{0},Bv\rangle=\langle F,v\rangle\ \ \forall\ v\in H,$$

$$\langle\eta,Bu^{0}\rangle=0\ \ \forall\ \eta\in V^{*},$$

respectively. Subtracting the second equation from the first one, and taking $v = u^{\epsilon} - u^{0}$, $\eta = \xi^{\epsilon} - \xi^{0}$, we get

$$(Au^{\epsilon}-Au^{0},Au^{\epsilon}-Au^{0})_{U}+\epsilon^{2}(\xi^{\epsilon}-\xi^{0},\xi^{\epsilon}-\xi^{0})_{V^{*}}=-\,\epsilon^{2}(\xi^{0},\xi^{\epsilon}-\xi^{0})_{V^{*}}.$$

By using Cauchy's inequality, we get

$$\|Au^{\epsilon}-Au^{0}\|_{U}^{2}+\epsilon^{2}\,\|\xi^{\epsilon}-\xi^{0}\|_{V^{*}}^{2}\lesssim\epsilon^{2}\,\|\xi^{0}\|_{V^{*}}^{2}.$$

The estimate (3.3.7) then follows from the fact that $\xi^{\epsilon} = \epsilon^{-2}\,\pi_{V}Bu^{\epsilon}$.

Now we assume $f|_{\ker B} = 0$. The variational problem (3.2.3) can be written as

$$\epsilon^2[(Au^\epsilon, Av)_U + (Bu^\epsilon, Bv)_V] + (1-\epsilon^2)(Bu^\epsilon, Bv)_V = \langle f, v\rangle \quad \forall\, v \in H. \tag{3.3.10}$$

By the equivalency assumption (3.2.1), the bilinear form

$$(u, v)_{\mathcal{H}} = (Au, Av)_U + (Bu, Bv)_V$$

defines an inner product on H, which is equivalent to the original inner product. With this new inner product, the space H will be denoted by $\mathcal{H}$. The condition $f|_{\ker B} = 0$ means that there exists a unique $u^0 \in (\ker B)^\perp$, the orthogonal complement of $\ker B$ in $\mathcal{H}$, such that

$$\langle f, v\rangle = (Bu^0, Bv)_V \;\; \forall\, v \in \mathcal{H}, \tag{3.3.11}$$

and the operator B defines an isomorphism between $(\ker B)^\perp$ and V.

From the equation (3.3.10), it is not hard to see that $u^\epsilon \in (\ker B)^\perp$. Substituting (3.3.11) into (3.3.10), and taking $v = u^\epsilon - u^0$, with a little algebra, we get

$$\begin{aligned} &\epsilon^2(u^\epsilon - u^0, u^\epsilon - u^0)_{\mathcal{H}} + (1-\epsilon^2)(Bu^\epsilon - Bu^0, Bu^\epsilon - Bu^0)_V \\ &\quad = \epsilon^2(Bu^0, Bu^\epsilon - Bu^0)_V - \epsilon^2(u^0, u^\epsilon - u^0)_{\mathcal{H}}. \end{aligned} \tag{3.3.12}$$

Since u^ϵ and u^0 both belong to $(\ker B)^\perp$, we have $\|u^0\|_{\mathcal{H}} \lesssim \|Bu^0\|_V$ and $\|u^\epsilon - u^0\|_{\mathcal{H}} \lesssim \|Bu^\epsilon - Bu^0\|_V$. Therefore, by using Cauchy's inequality, from (3.3.12), we get

$$\epsilon^2(u^\epsilon - u^0, u^\epsilon - u^0)_{\mathcal{H}} + (1-\epsilon^2)(Bu^\epsilon - Bu^0, Bu^\epsilon - Bu^0)_V \lesssim \epsilon^4 \|Bu^0\|_V^2. \quad (3.3.13)$$

Therefore,

$$\|u^\epsilon - u^0\|_{\mathcal{H}} \simeq \|Bu^\epsilon - Bu^0\|_V \lesssim \epsilon^2 \|Bu^0\|_V = \epsilon^2 \|\zeta^0\|_V.$$

The estimate (3.3.9) is a consequence of the equivalence assumption (3.2.1). □

To get the estimate (3.3.8), the assumption that the operator B has closed range is crucial. Without this assumption, we can not expect the convergence of the sequence $\{u^\epsilon\}$ in the space H. This is a usual phenomena in singular perturbation problems and a big trouble for numerical analysis of the general membrane–shear shells. This result is the infinite dimensional version of the so-called Cheshire lemma.

The condition that the operator B is surjective is only satisfied by some special problems. This condition is not met by the Reissner–Mindlin plate bending model, nor the Koiter or Naghdi shell models. It does not apply to the spherical shell model and the general shell model we are going to derive and justify either.

If the range of B is not closed, the space W is a proper subspace of V, so is V^* of W^*. If V^* is identified with V through the Riesz representation theorem, we have the inclusions $W \subset V \sim V^* \subset W^*$. We will show that in the case of $f|_{\ker B} \neq 0$, the asymptotic behavior of the solution u^ϵ is largely determined by the position of ξ^0, the Lagrange multiplier defined in (3.3.4), between V^* and W^*. The closer ξ^0 to V^*, the stronger the convergence. In the best case of $\xi^0 \in V^*$, a convergence rate of the form

(3.3.7) can be obtained. In the worst case, i.e., we only have $\xi^0 \in W^*$, we will prove a convergence, but without convergence rate.

In the case of $f|_{\ker B} = 0$, we must require ζ_*^0, the equivalent representation of f in W^* defined in (3.3.6), to be in the smaller space V^*. Then, the asymptotic behavior of u^ϵ is determined by the position of ζ^0, the Riesz representation of ζ_*^0 in the space V, between W and V.

3.3.2 The case of flexural domination

In this subsection, we discuss the case of $f|_{\ker B} \neq 0$ without the assumption that B is surjective. In this case we need to rescale the problem by assuming that $F = \epsilon^{-2} f$ is independent of ϵ. We have

THEOREM 3.3.2. *Let $(u^\epsilon, \xi^\epsilon)$ be the solution of (3.2.6), and (u^0, ξ^0) be the solution of (3.3.4), then*

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1}\|Bu^\epsilon\|_V \lesssim \epsilon \|\xi^0\|_{\epsilon^{-1} W^* + V^*} \tag{3.3.14}$$

Proof. Subtracting (3.3.4) from (3.2.6), we get

$$\begin{aligned} (A(u^\epsilon - u^0), Av)_U + \langle \xi^\epsilon - \xi^0, Bv\rangle = 0 \ \ \forall\, v \in H, \\ \langle B(u^\epsilon - u^0), \eta\rangle - \epsilon^2(\xi^\epsilon, \eta)_{V^*} = 0 \ \ \forall\, \eta \in V^*. \end{aligned} \tag{3.3.15}$$

Taking $v = u^\epsilon - u^0$ and $\eta = \xi^\epsilon$, and writing the second equation as

$$\langle B(u^\epsilon - u^0), \xi^\epsilon - \xi^0\rangle + \langle B(u^\epsilon - u^0), \xi^0\rangle - \epsilon^2(\xi^\epsilon, \xi^\epsilon)_{V^*} = 0,$$

together with the first equation

$$(A(u^\epsilon - u^0), A(u^\epsilon - u^0))_U + \langle \xi^\epsilon - \xi^0, B(u^\epsilon - u^0)\rangle = 0,$$

we get

$$(A(u^\epsilon - u^0), A(u^\epsilon - u^0))_U + \epsilon^2 (\xi^\epsilon, \xi^\epsilon)_{V^*} = \langle B(u^\epsilon - u^0), \xi^0\rangle. \tag{3.3.16}$$

By the definition (3.2.5) of the norm of W, and the equivalence assumption (3.2.1), we have

$$\|B(u^\epsilon - u^0)\|_W \lesssim \|A(u^\epsilon - u^0)\|_U + \|B(u^\epsilon - u^0)\|_V,$$

and so

$$\|B(u^\epsilon - u^0)\|_{\epsilon W} \lesssim \epsilon \, \|A(u^\epsilon - u^0)\|_U + \epsilon \, \|B(u^\epsilon - u^0)\|_V.$$

Therefore, we have the estimate

$$|\langle B(u^\epsilon - u^0), \xi^0\rangle| \lesssim [\epsilon \, \|A(u^\epsilon - u^0)\|_U + \|B(u^\epsilon - u^0)\|_V] \|\xi^0\|_{\epsilon^{-1} W^* + V^*}. \tag{3.3.17}$$

Recalling that $\xi^\epsilon = \pi_V \, \epsilon^{-2} \, Bu^\epsilon$ and $Bu^0 = 0$, combining (3.3.16) and (3.3.17) and using Cauchy's inequality, we obtain

$$\|A(u^\epsilon - u^0)\|_U^2 + \epsilon^{-2} \, \|Bu^\epsilon\|_V^2 \lesssim \epsilon^2 \, \|\xi^0\|_{\epsilon^{-1} W^* + V^*}^2.$$

The desired result then follows. □

The K-functional on the Hilbert couple $[W^*, V^*]$ is given by

$$K(\epsilon, \xi^0, [W^*, V^*]) = \epsilon \|\xi^0\|_{\epsilon^{-1} W^* + V^*},$$

see [9]. According to the definition of interpolation spaces based on the K-functional,

$$|K(\epsilon, \xi^0, [W^*, V^*])| \lesssim C_{\theta,q}\, \epsilon^\theta \|\xi^0\|_{[V^*,W^*]_{1-\theta,q}}.$$

If ξ^0 is further assumed to belong to the interpolation space $[V^*, W^*]_{1-\theta,q}$, for some $0 < \theta < 1$ and $1 \leq q \leq \infty$, or $0 \leq \theta \leq 1$ and $1 < q < \infty$, we have

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V \lesssim \epsilon^\theta \|\xi^0\|_{[V^*,W^*]_{1-\theta,q}}. \tag{3.3.18}$$

In particular, if

$$\xi^0 \in V^*, \tag{3.3.19}$$

we can take $\theta = 1$ and obtain the stronger result

$$\|Au^\epsilon - Au^0\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V \lesssim \epsilon \|\xi^0\|_{V^*}. \tag{3.3.20}$$

The estimate (3.3.20) is an extension of a convergence theorem of solution of the Reissner–Mindlin plate bending model to that of the Kirchhoff–Love plate bending model in [4].

We will prove that the convergence rate of our 2D shell model solution to the 3D shell solution in the relative energy norm is crucially related to this “regularity index” θ of the Lagrange multiplier ξ^0.

The condition (3.3.19) can be verified for the Reissner–Mindlin plate bending model, if the plate is totally clamped, and the plate boundary and loading function are smooth enough so that the H^3 regularity of the Kirchhoff–Love model solution holds, see [8]. For partially clamped plates and arbitrary shells, this index needs to be carefully evaluated.

If we know nothing more than the minimum regularity of the Lagrange multiplier $\xi^0 \in W^*$, then we must choose $\theta = 0$. The estimate (3.3.18) does not provide any useful information. The following theorem will be used to prove the convergence, but without a convergence rate, of the flexural shell model.

Theorem 3.3.3. *Let u^ϵ be the solution of (3.2.6) and u^0 be the solution of (3.3.4). We have the convergence result*

$$\lim_{\epsilon \longrightarrow 0} [\|A(u^\epsilon - u^0)\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V] = 0. \tag{3.3.21}$$

Proof. Taking $v = u^\epsilon$ in the equation (3.2.3), we get

$$(Au^\epsilon, Au^\epsilon)_U + \epsilon^{-2}(Bu^\epsilon, Bu^\epsilon)_V = \langle F, u^\epsilon \rangle.$$

From this equation we see that there exists a constant C independent of ϵ, such that

$$\|u^\epsilon\|_H + \|\,\epsilon^{-1}\,Bu^\epsilon\|_V \le C.$$

Since bounded sets in a Hilbert space are weakly compact, there exists a subsequence, $\epsilon_n \to 0$, an element $\tilde{u} \in H$, and an element $p \in V$, such that

$$u^{\epsilon_n} \rightharpoonup \tilde{u} \text{ in } \ H, \quad \epsilon_n^{-1}\,Bu^{\epsilon_n} \rightharpoonup p \ \text{ in } \ V.$$

Since $Bu^{\epsilon_n} \rightharpoonup B\tilde{u}$ in V, and $Bu^{\epsilon_n} \to 0$ in V, we have $B\tilde{u} = 0$, so $\tilde{u} \in \ker B$. Note that we also have $Au^{\epsilon_n} \rightharpoonup A\tilde{u}$, so we have

$$(A\tilde{u}, Av)_U = \langle F, v\rangle \quad \forall\ v \in \ker B,$$

so $\tilde{u} = u^0$, the solution of (3.3.3). Therefore the whole sequence $\{u^\epsilon\}$ weakly converges to u^0 in H.

Consider the identity

$$(Au^\epsilon - Au^0, Au^\epsilon - Au^0)_U + (\epsilon^{-1}\,Bu^\epsilon - p, \epsilon^{-1}\,Bu^\epsilon - p)_V$$

$$= (Au^\epsilon, Au^\epsilon)_U + \epsilon^{-2}(Bu^\epsilon, Bu^\epsilon)_V + (Au^0, Au^0 - 2Au^\epsilon)_U + (p, p - 2\,\epsilon^{-1}\,Bu^\epsilon)_V$$

$$= \langle F, u^\epsilon\rangle + (Au^0, Au^0 - 2Au^\epsilon)_U + (p, p - 2\,\epsilon^{-1}\,Bu^\epsilon)_V. \quad (3.3.22)$$

For the above subsequence, the right hand side, as a sequence of numbers, converges to

$$\langle F, u^0 \rangle - (Au^0, Au^0)_U - (p, p)_V = -(p, p)_V,$$

while the left hand side of (3.3.22) is nonnegative, so we must have $p = 0$. Therefore the whole sequence $\epsilon^{-1} Bu^\epsilon$ weakly converges to zero. The desired strong convergence follows from the same identity. □

This theorem is an extension of, and its proof was adapted from [4] for Reissner–Mindlin plate bending, [21] for flexural Naghdi shell, and [15] for flexural Koiter shell problems.

This theorem shows that if $f|_{\ker B} \neq 0$, the problem is bending or flexural dominated in the sense of

$$\frac{(Bu^\epsilon, Bu^\epsilon)_V}{\epsilon^2 (Au^\epsilon, Au^\epsilon)_U} \to 0 \quad (\epsilon \to 0).$$

3.3.3 The case of membrane–shear domination

If $f|_{\ker B} = 0$, the solution of the limiting problem (3.3.3) will be zero. If we still assume $F = \epsilon^{-2} f$ is independent of ϵ, the above estimate only gives the following convergence.

$$\|Au^\epsilon\|_U + \epsilon^{-1} \|Bu^\epsilon\|_V \to 0 \quad (\epsilon \to 0). \tag{3.3.23}$$

This convergence will be useful when we analyze the relationship between our theory and other shell theories. It is also needed to resolve the effect of the higher order term $\epsilon^2 f_1$ in the loading functional. Otherwise, it hardly tells us more than that the solution u^ϵ converges to zero, and fails to fully capture the asymptotic behavior of the solution. To get a good grasp of the asymptotic behavior of the solution, we will assume that f is independent of ϵ in this case. In this case, there exists a unique $\zeta_*^0 \in W^*$ such that

$$\langle f, v\rangle = \langle \zeta_*^0, Bv\rangle \quad \forall\, v \in H.$$

Equivalently,

$$\langle F, v\rangle = \langle \epsilon^{-2}\, \zeta_*^0, Bv\rangle \quad \forall\, v \in H.$$

Without further assumption on ζ_*^0, we can not get any useful results for our model justification. We will derive an estimate under the assumption

$$\zeta_*^0 \in V^*, \tag{3.3.24}$$

so $\zeta^0 = i_V \zeta_*^0$ is well-defined. This condition does exclude some physically meaningful shell problems. However, if this condition is not satisfied, the 2D model solution, very likely, does not approximate the 3D elasticity solution in the energy norm.

The mixed problem (3.2.6) may be written as

$$(Au, Av)_U + \langle \xi, Bv\rangle = \langle \epsilon^{-2}\, \zeta_*^0, Bv\rangle \quad \forall\, v \in H,$$

$$\langle \eta, Bu\rangle - \epsilon^2 (\xi, \eta)_{V^*} = 0 \quad \forall\, \eta \in V^*,$$

$$u \in H, \quad \xi \in V^*.$$

Under the assumption (3.3.24), this problem can be rewritten as

$$(Au, Av)_U + \langle \xi - \epsilon^{-2}\, \zeta_*^0, Bv\rangle = 0 \quad \forall\, v \in H,$$

$$\langle \eta, Bu\rangle - \epsilon^2 (\xi - \epsilon^{-2}\, \zeta_*^0, \eta)_{V^*} = (\zeta_*^0, \eta)_{V^*} = \langle \zeta^0, \eta\rangle \tag{3.3.25}$$

$$\forall\, \eta \in V^*,\; u \in H, \quad \xi \in V^*.$$

This formulation is in the form of our general mixed problem (3.2.8). Therefore, by Theorem 3.2.3, we have the equivalence

$$\|u^\epsilon\|_H + \|\xi^\epsilon - \epsilon^{-2}\, \zeta_*^0\|_{W^* \cap \epsilon\, V^*} \simeq \|\zeta^0\|_{W + \epsilon^{-1}\, V}. \tag{3.3.26}$$

Recalling that $\xi^\epsilon = \epsilon^{-2}\, \pi_V B u^\epsilon$, we get the equivalence

$$\|u^\epsilon\|_H + \epsilon^{-2}\, \|\pi_V B u^\epsilon - \zeta_*^0\|_{W^* \cap \epsilon\, V^*} \simeq \|\zeta^0\|_{W + \epsilon^{-1}\, V}. \tag{3.3.27}$$

Therefore,

$$\|u^\epsilon\|_H + \epsilon^{-2}\, \|\pi_V B u^\epsilon - \zeta_*^0\|_{W^*} + \epsilon^{-1}\, \|B u^\epsilon - \zeta^0\|_V \simeq \|\zeta^0\|_{W + \epsilon^{-1}\, V}.$$

In particular, we have proved the following theorem.

THEOREM 3.3.4. *In the case of* $f|_{\ker B} = 0$, *and under the assumption* $\zeta^0_* \in V^*$, *we have the following estimates:*

$$\begin{aligned} \epsilon \, \|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V &\lesssim \epsilon \, \|\zeta^0\|_{W+\epsilon^{-1}V}, \\ \|\pi_V Bu^\epsilon - \zeta^0_*\|_{W^*} &\lesssim \epsilon^2 \, \|\zeta^0\|_{W+\epsilon^{-1}V}. \end{aligned} \tag{3.3.28}$$

In terms of the K-functional, we have

$$\epsilon \, \|\zeta^0\|_{W+\epsilon^{-1}V} = K(\epsilon, \zeta^0, [V, W]).$$

If ζ^0 belongs to the interpolation space $[W, V]_{1-\theta,q}$ for some $0 < \theta < 1$ and $1 \le q \le \infty$, or $0 \le \theta \le 1$ and $1 < q < \infty$, we have

$$\epsilon \, \|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V \lesssim \epsilon^\theta \, \|\zeta^0\|_{[W,V]_{1-\theta,q}}. \tag{3.3.29}$$

In particular, if $\zeta^0 \in W$, we can take $\theta = 1$ and obtain

$$\epsilon \, \|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V \lesssim \epsilon \, \|\zeta^0\|_W. \tag{3.3.30}$$

The "regularity index" θ of ζ^0, i.e., the largest θ such that $\zeta^0 \in [W, V]_{1-\theta,q}$, which will determine the convergence rate of the shell model in the relative energy norm, can be attributed to the regularity of the shell data, but generally it is hard to interpret in terms of smoothness in the Sobolev sense. For a totally clamped elliptic shell, we can show that $\theta = 1/6$ under smoothness assumptions on the shell boundary and loading functions in the usual sense. For the shear dominated Reissner–Mindlin plate bending, reasonable assumptions on the smoothness of the loading functions will lead to $\theta = 1/2$.

If we only have the minimum regularity assumption $\zeta^0 \in V$, we just have $\theta = 0$, and the estimate (3.3.29) reduces to

$$\epsilon \|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V \lesssim \|\zeta^0\|_V, \tag{3.3.31}$$

which does not tell anything useful. We can construct example to show that this estimation is optimal. Due to the ϵ independence of f, we have the strong convergence stated in the next theorem. This convergence will be used to prove the convergence of the model, although without a convergence rate.

THEOREM 3.3.5. *If $f|_{\ker B} = 0$, and its representative $\zeta_*^0 \in V^*$, we have the strong convergence*

$$\lim_{\epsilon \to 0} [\epsilon \|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V] = 0.$$

Proof. From (3.3.31), we see that there exist a constant C independent of ϵ, such that

$$\| \epsilon Au^\epsilon\|_U \le C, \quad \|Bu^\epsilon\|_V \le C.$$

Therefore, there exist a subsequence, $\epsilon_n \to 0$, an element $p \in U$, and an element $v^0 \in V$, such that $\epsilon_n Au^{\epsilon_n} \rightharpoonup p$ in U, and $Bu^{\epsilon_n} \rightharpoonup v^0$ in V. Since

$$\epsilon_n^2 (Au^{\epsilon_n}, Av)_U + (Bu^{\epsilon_n}, Bv)_V = (\zeta^0, Bv)_V \quad \forall v \in V,$$

we have $Bu^{\epsilon_n} \rightharpoonup \zeta^0$ in V, therefore $v^0 = \zeta^0$.

The following identity can be verified:

$$
\begin{aligned}
&(\epsilon\, Au^\epsilon - p, \epsilon\, Au^\epsilon - p)_U + (Bu^\epsilon - \zeta^0, Bu^\epsilon - \zeta^0)_V \\
&\quad = \epsilon^2 (Au^\epsilon, Au^\epsilon)_U + (Bu^\epsilon, Bu^\epsilon)_V + (p, p - 2\,\epsilon\, Au^\epsilon)_U + (\zeta^0, \zeta^0 - 2Bu^\epsilon)_V \\
&\quad = (\zeta^0, Bu^\epsilon)_V + (p, p - 2\,\epsilon\, Au^\epsilon)_U + (\zeta^0, \zeta^0 - 2Bu^\epsilon)_V. \qquad (3.3.32)
\end{aligned}
$$

When applied to the subsequence $\{u^{\epsilon_n}\}$, the right hand side of this identity converges to

$$
(\zeta^0, \zeta^0)_V - (p, p)_U - (\zeta^0, \zeta^0)_V = -(p, p)_U.
$$

Since the left-hand side is nonnegative, we must have $p = 0$. Therefore, the whole sequence $\epsilon\, Au^\epsilon$ weakly converges to zero, and the whole sequence Bu^ϵ weakly converges to ζ^0. The strong convergence follows from the above identity. □

This proof was adapted from [36] for singular perturbation problems, [21] for membrane Koiter shell, and [15] for membrane Naghdi shell.

Under the condition of this theorem, the problem is membrane–shear dominated in the sense of

$$
\frac{\epsilon^2 (Au^\epsilon, Au^\epsilon)_U}{(Bu^\epsilon, Bu^\epsilon)_V} \to 0 \quad (\epsilon \to 0).
$$

The above analysis shows that if $f|_{\ker B} = 0$ and $\zeta^0_* \in V^*$, or more informatively, $\zeta^0 \in [W, V]$, we have membrane–shear domination. If ζ^0_* does not belong to V^*, the behavior of the solution can be very complicated. Usually, there is no membrane-shear domination, but rather, the flexural energy $\epsilon^2(Au^\epsilon, Au^\epsilon)_U$ might be comparable to the membrane–shear energy $(Bu^\epsilon, Bu^\epsilon)_V$, see [12] and [41], although the geometry of its middle surface and the type of the boundary conditions may classify a shell as a membrane shell. For example, a partially clamped elliptic shell may behave this way even for infinitely differentiable loading functions. In this case, the limiting membrane shell model has no solution. Although our model provides a solution, we are not able to justify it.

The following corollary to Theorems 3.3.4 and 3.3.5 will be used when we construct corrections for the transverse deflection, which are necessary for the convergence of the shell model in the relative energy norm.

THEOREM 3.3.6. *Let $\omega \subset \mathbb{R}^2$ be a bounded, connected open domain whose boundary is partitioned as $\partial\omega = \partial_D\omega \cup \partial_T\omega$. The function space H^1_D is a subspace of H^1 whose elements vanish on $\partial_D\omega$. The variational problem*

$$\epsilon^2(\nabla u^\epsilon, \nabla v)_{\underset{\sim}{L_2}} + (u^\epsilon, v)_{L_2} = \langle f, v\rangle \quad \forall v \in H^1_D, \quad u^\epsilon \in H^1_D \tag{3.3.33}$$

has a unique solution $u^\epsilon \in H^1_D$ for any $f \in {H^1_D}^*$. If $f \in L_2$, we have the estimate

$$\epsilon \, \|\nabla u^\epsilon\|_{\underset{\sim}{L_2}} + \|u^\epsilon - f\|_{L_2} \lesssim \epsilon \, \|f\|_{H^1_D + \epsilon^{-1} L_2}. \tag{3.3.34}$$

If $f \in H^1$, the standard cut-off argument gives

$$\epsilon \, \|\nabla u^\epsilon\|_{\underset{\sim}{L_2}} + \|u^\epsilon - f\|_{L_2} \lesssim \epsilon^{1/2} \, \|f\|_{H^1}. \tag{3.3.35}$$

If we assume that the interpolation norm $\|f\|_{[H^1_D, L_2]_{1-\theta,q}}$ *is finite for some* $\theta \in (0,1)$ *and* $q \in [1,\infty]$*, or* $\theta \in [0,1]$ *and* $q \in (1,\infty)$*, we have*

$$\epsilon \|\nabla u^\epsilon\|_{\underset{\sim}{L}_2} + \|u^\epsilon - f\|_{L_2} \lesssim \epsilon^\theta \|f\|_{[H^1_D, L_2]_{1-\theta,q}}. \tag{3.3.36}$$

In particular, if $f \in H^1_D$*, we have*

$$\epsilon \|\nabla u^\epsilon\|_{\underset{\sim}{L}_2} + \|u^\epsilon - f\|_{L_2} \lesssim \epsilon \|f\|_{H^1}. \tag{3.3.37}$$

If we only know that $f \in L_2$*, the strong convergence*

$$\lim_{\epsilon \to 0} [\epsilon \|\nabla u^\epsilon\|_{\underset{\sim}{L}_2} + \|u^\epsilon - f\|_{L_2}] = 0 \tag{3.3.38}$$

holds.

Proof. The conclusions follow from the above theorems by letting $H = H^1_D$, $U = \underset{\sim}{L}_2$, $V = L_2$, $A = \nabla$, and $B =$ identity. □

A direct proof of (3.3.35) can be found in [2].

3.4 Parameter-dependent loading functional

In this section, we discuss the behavior of the solution of the variational problem

$$\begin{gathered} \epsilon^2 (Au, Av)_U + (Bu, Bv)_V = \langle f_0 + \epsilon^2 f_1, v \rangle, \\ u \in H, \quad \forall\, v \in H, \end{gathered} \tag{3.4.1}$$

in which both f_0 and f_1 are independent of ϵ, and $f_0 \neq 0$. This is a problem abstracted from our shell models. This form of the resultant loading functional is a consequence of our assumption on the loading functions. To grasp the behavior of solution of the problem with such ϵ-dependent loading functionals, we apply the above theory to the problems whose right hand sides are f_0 and $\epsilon^2 f_1$ respectively. The desired behavior will be obtained by superposition.

Let $f = f_0$ in the equations and theorems in the previous two sections, and consider the problem

$$\begin{gathered} \epsilon^2 (Au_1, Av)_U + (Bu_1, Bv)_V = \epsilon^2 \langle f_1, v \rangle, \\ u_1 \in H, \quad \forall\, v \in H, \end{gathered} \tag{3.4.2}$$

which is due to the higher order term in the loading functional. This problem has a unique solution u_1^ϵ, and by Theorem 3.3.3, we have

$$\lim_{\epsilon \to 0} [\|A(u_1^\epsilon - u_1^0)\|_U + \epsilon^{-1} \, \|Bu_1^\epsilon\|_V] = 0, \tag{3.4.3}$$

where $u_1^0 \in H$ is defined as the solution of (3.3.3) or (3.3.4) with F replaced by f_1. Note that u_1^0 may be zero or nonzero depending on whether $f_1|_{\ker B} = 0$ or not.

We will see that the problem should be classified by the leading term f_0, and we discuss the problem separately for whether or not $f_0|_{\ker B} \neq 0$.

If $f_0|_{\ker B} \neq 0$, we need to scale the loading functionals f_0 and $\epsilon^2 f_1$ simultaneously. The solution of (3.2.2) will be given by $\tilde{u}^\epsilon = u^\epsilon + \epsilon^2 \, u_1^\epsilon$ with u^ϵ and u_1^ϵ the solutions of (3.2.6) and (3.4.2) respectively. Under the condition of Theorem 3.3.2, by (3.3.18) together with (3.4.3) we get the estimate

$$\|A\tilde{u}^\epsilon - Au^0\|_U + \epsilon^{-1}\|B\tilde{u}^\epsilon\|_V$$

$$\lesssim \|Au^\epsilon - Au^0\|_U + \epsilon^{-1}\|Bu^\epsilon\|_V + \epsilon^2(\|Au_1^\epsilon\|_U + \epsilon^{-1}\|Bu_1^\epsilon\|_V)$$

$$\lesssim \epsilon^\theta\|\xi^0\|_{[V^*,W^*]_{1-\theta,q}} + O(\epsilon^2). \quad (3.4.4)$$

Under the condition of Theorem 3.3.3, we have

$$\|A\tilde{u}^\epsilon - Au^0\|_U + \epsilon^{-1}\|B\tilde{u}^\epsilon\|_V \lesssim o(1) + O(\epsilon^2). \quad (3.4.5)$$

Therefore, in the case of flexural shells, adding the higher order term $\epsilon^2 f_1$ to the right hand side does not disturb the asymptotic behavior of the solution of (3.2.2) determined by the leading term.

If $f|_{\ker B} = 0$, there is no need to scale the loading functional and the solution of (3.2.2) is given by $\tilde{u}^\epsilon = u^\epsilon + u_1^\epsilon$ with u^ϵ and u_1^ϵ defined as solutions of (3.2.3) and (3.4.2) respectively. Under the condition of Theorem 3.3.4, corresponding to the convergence (3.3.29), by using (3.4.3), we have

$$\epsilon\|A\tilde{u}^\epsilon\|_U + \|B\tilde{u}^\epsilon - \zeta^0\|_V$$

$$\lesssim \epsilon\|Au^\epsilon\|_U + \|Bu^\epsilon - \zeta^0\|_V + \epsilon(\|Au_1^\epsilon\|_U + \epsilon^{-1}\|Bu_1^\epsilon\|_V)$$

$$\lesssim \epsilon^\theta\|\zeta^0\|_{[W,V]_{1-\theta,q}} + O(\epsilon)[o(\epsilon)], \quad (3.4.6)$$

if $u_1^0 \neq 0[= 0]$. Under the condition of Theorem 3.3.5, we have

$$\epsilon \|A\tilde{u}^\epsilon\|_U + \|B\tilde{u}^\epsilon - \zeta^0\|_V \lesssim o(1) + O(\epsilon)[o(\epsilon)], \tag{3.4.7}$$

if $u_1^0 \neq 0[= 0]$. Therefore, the higher order term $\epsilon^2 f_1$ does not affect the asymptotic behaviors of the solution of (3.2.2), as described by Theorems 3.3.4 and 3.3.5.

If the range of B is closed, or in the situation of Theorem 3.3.1, the additional term $\epsilon^2 f_1$ does not affect the asymptotic behaviors described by (3.3.7) and (3.3.9). But the stronger convergence (3.3.8) will be affected, especially when $f|_{\ker B} = 0$ and $f_1|_{\ker B} \neq 0$. In this case the contribution to the solution from $\epsilon^2 f_1$ will be finite, by correcting the limit, we will get a convergence in the form of (3.3.8) while the convergence rate needs to be reduced from ϵ^2 to ϵ, see (2.5.4) and (2.5.7).

In Chapter 7, we will discuss the the model under the usual assumption on the loading functions. For that purpose, we need to consider the problem

$$\begin{gathered} \epsilon^2 (Au, Av)_U + (Bu, Bv)_V = \langle f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3, v\rangle, \\ u \in H, \quad \forall\, v \in H, \end{gathered} \tag{3.4.8}$$

with f_0, f_1, f_2, and f_3 independent of ϵ, and $f_0 \neq 0$. The theory can be applied to problems of the form (3.2.3), with right hand sides f_0, ϵf_1, $\epsilon^2 f_2$, and $\epsilon^3 f_3$, respectively. The desired behavior will be obtained by superposition. Since we will not discuss the convergence rate of other shell theories in details in this thesis, we will not list the results corresponding to (3.4.4) and (3.4.6). We still denote the solution of (3.4.8) by $\tilde{u}^\epsilon$. The following convergence results can be obtained.

If $f_0|_{\ker B} \neq 0$, we have

$$\|A\tilde{u}^\epsilon - Au^0\|_U + \epsilon^{-1}\|B\tilde{u}^\epsilon\|_V \lesssim o(1). \tag{3.4.9}$$

If $f_0|_{\ker B} = 0$, $f_1|_{\ker B} = 0$, under the condition of theorem 3.3.5 we have

$$\epsilon\,\|A\tilde{u}^\epsilon\|_U + \|B\tilde{u}^\epsilon - \zeta^0\|_V \lesssim o(1). \tag{3.4.10}$$

The asymptotic behavior of solution of (3.4.1) was not harmed by adding $\epsilon\, f_1 + \epsilon^2\, f_2 + \epsilon^3\, f_3$ to the loading functional.

If $f_0|_{\ker B} = 0$ but $f_1|_{\ker B} \neq 0$, we only have

$$\epsilon\,\|A\tilde{u}^\epsilon\|_U + \|B\tilde{u}^\epsilon - \zeta^0\|_V \lesssim O(1). \tag{3.4.11}$$

In this case, the expected membrane-shear dominated asymptotic behavior described by Theorem 3.3.5 was severely affected by adding the higher order term $\epsilon\, f_1$ to the loading functional. This is a rare situation in which the leading term f_0 puts the problem in the category of membrane-shear shells, while the higher order term $\epsilon\, f_1$ draws it into the category of flexural shell. In the sum, neither of them can dominate.

3.5 Classification

For the abstract variational problem

$$\epsilon^2(Au^\epsilon, Av)_U + (Bu^\epsilon, Bv)_V = \langle f_0 + \epsilon^2 f_1, v\rangle,$$

$$u^\epsilon \in H, \quad \forall\, v \in H,$$

the two estimates (3.4.4) and (3.4.5) show that if $f_0|_{\ker B} \neq 0$, the flexural energy $\epsilon^2(Au^\epsilon, Au^\epsilon)_U$ dominates. In this case, the shell problem will be called a flexural shell.

The two estimates (3.4.6) and (3.4.7) show that when $f_0|_{\ker B} = 0$ and its representation $\zeta^0_* \in V^*$, the membrane–shear energy $(Bu^\epsilon, Bu^\epsilon)_V$ dominates. In this case, the shell problem will be called a membrane–shear shell. A membrane–shear shell will be called a first kind membrane–shear shell or stiff membrane–shear shell if $\ker B = 0$. If $\ker B \neq 0$ but $f_0|_{\ker B} = 0$, the shell will be called a second kind membrane–shear shell.

We will justify the shell model in both of the above cases. If $f_0|_{\ker B} = 0$, but ζ^0_* does not belong to V^*, the shell model can not be justified.

Chapter 4

Three-dimensional shells

In this chapter, we briefly review the linearized 3D elasticity theory for a thin elastic shell in the curvilinear coordinates and recall all the materials from the differential geometry of surfaces that will be necessary for the shell analyses. Special curvilinear coordinates on 3D shells, which are attached to coordinates on the middle surfaces, will be defined. Rescaled stress components, rescaled applied force components, and rescaled displacement components will be introduced. In terms of the rescaled components, the linearized elasticity equations have a noticeably simpler form, and calculations can be substantially simplified. We also recall the two energies principle that will be the fundamental tool for our justification of the spherical shell model.

4.1 Curvilinear coordinates on a shell

Let $\omega \subset \mathbb{R}^2$ be a bounded connected open domain, whose boundary $\partial\omega$ is smooth. We use $\underset{\sim}{x} = (x_1, x_2)$ to denote the Cartesian coordinates of a generic point in $\bar{\omega}$. A surface $S \subset \mathbb{R}^3$ is defined as the image of the set $\bar{\omega}$ through a mapping $\boldsymbol{\phi}$ from $\bar{\omega}$ to $\mathbb{R}^3$. We assume that the mapping is injective and fairly smooth. The boundary of S is $\gamma = \boldsymbol{\phi}(\partial\omega)$. The pair of numbers $\underset{\sim}{x} = (x_1, x_2)$ then furnishes the curvilinear coordinates on the surface S. We assume that at any point on the surface, along the coordinate lines, the two tangential vectors $\boldsymbol{a}_\alpha = \partial\boldsymbol{\phi}/\partial x_\alpha$ are linearly independent. The unit vector $\boldsymbol{a}_3$

that is normal to the surface can be expressed as

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$

At any point on the surface, the three vectors $\boldsymbol{a}_i$ furnish the covariant basis. The contravariant basis $\boldsymbol{a}^i$ is defined by the relations $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta$ and $\boldsymbol{a}^3 = \boldsymbol{a}_3$, in which δ^α_β is the Kronecker delta. It is obvious that $\boldsymbol{a}^\alpha$ are also tangent to the surface.

The first fundamental form on the surface, or the metric tensor, $a_{\alpha\beta}$ is defined by $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$, which is symmetric positive definite. The contravariant components of the metric tensor are given by $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$.

The second fundamental form, or the curvature tensor, $b_{\alpha\beta}$ is defined by $b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_\alpha$, which is also symmetric. The mixed curvature tensor is $b^\alpha_\beta = a^{\alpha\gamma} b_{\gamma\beta}$. The tensor $c_{\alpha\beta} = b^\gamma_\alpha b_{\gamma\beta}$ is called the third fundamental form, which is also symmetric.

The trace and determinant of the mixed curvature tensor b^α_β (as a matrix) are intrinsic quantities of the surface which are independent of the coordinates. They are the mean curvature and Gauss curvature respectively, denoted by

$$H = \frac{1}{2}(b^1_1 + b^2_2) \quad \text{and} \quad K = b^1_1 b^2_2 - b^1_2 b^2_1.$$

The three fundamental forms and the two curvatures are connected by the identity

$$K a_{\alpha\beta} - 2H b_{\alpha\beta} + c_{\alpha\beta} = 0.$$

Expressed in mixed components of the tensors, this identity easily follows from the Hamilton–Cayley theorem in matrix analysis.

The Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ are defined by $\Gamma^{\gamma}_{\alpha\beta} = \boldsymbol{a}^{\gamma} \cdot \partial_{\beta} \boldsymbol{a}_{\alpha}$, which are symmetric with respect to the subscripts, i.e., $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$.

The shell with middle surface S and thickness $2\,\epsilon$, is a 3D elastic body occupying the domain $\Omega^{\epsilon} \subset \mathbb{R}^3$, which is the image of the plate $\omega^{\epsilon} = \bar{\omega} \times [-\,\epsilon, \epsilon]$ through the mapping $\boldsymbol{\Phi}$:

$$\boldsymbol{\Phi}(x_1, x_2, t) = \boldsymbol{\phi}(x_1, x_2) + t\boldsymbol{a}^3, \quad (x_1, x_2) \in \bar{\omega}, \quad t \in [-\,\epsilon, \epsilon].$$

We assume that ϵ is small enough so that $\boldsymbol{\Phi}$ is injective. The triple of numbers (x_1, x_2, t) furnishes the curvilinear coordinates on the shell Ω^{ϵ}. We may use $t = x_3$ exchangeably for convenience. Corresponding to these curvilinear coordinates, the covariant basis vectors at any point in Ω^{ϵ} are defined by

$$\boldsymbol{g}_i(x_1, x_2, x_3) = \frac{\partial \boldsymbol{\Phi}(x_1, x_2, x_3)}{\partial x_i}.$$

The 3D second order tensor $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ is called the covariant metric tensor, whose determinant is denoted by $g = \det(g_{ij})$. The contravariant metric tensor g^{ij} is defined as the inverse of g_{ij} as a matrix, so, $g^{ik} g_{kj} = \delta^i_j$. The triple of vectors $\boldsymbol{g}^i = g^{ij} \boldsymbol{g}_j$ furnishes the contravariant basis. Note that $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$.

A vector field $\boldsymbol{v}$ can be given in terms of its covariant components v_i or contravariant components v^i through the relation $\boldsymbol{v} = v_i \boldsymbol{g}^i = v^i \boldsymbol{g}_i$. A tensor field $\boldsymbol{\sigma}$ can be given in terms of its contravariant components σ^{ij}, covariant components σ_{ij}, or mixed components σ^i_j through the relations

$$\boldsymbol{\sigma} = \sigma^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \sigma_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j = \sigma^i_j\boldsymbol{g}_i \otimes \boldsymbol{g}^j.$$

For brevity, we will use notations like $\boldsymbol{v} = v_i = v^i$ and $\boldsymbol{\sigma} = \sigma^{ij} = \sigma_{ij} = \sigma^i_j$. The covariant components of a tensor will be called a covariant tensor, etc.

The Christoffel symbols are defined by $\Gamma^{*k}_{ij} = \boldsymbol{g}^k \cdot \partial_j \boldsymbol{g}_i$. The superscript $*$ is added to indicate the difference from the Christoffel symbols on the middle surface. The indices of all tensors and the Christoffel symbols can be raised or lowered by multiplication and contraction with the contravariant or covariant metric tensors.

For any vector or tensor defined on the shell Ω^ϵ, we can define its covariant and contravariant derivatives, which themselves are tensors of higher orders. We use double vertical bar to denote the derivatives on the 3D shell. For example, the covariant derivative of the stress tensor σ^{ij} is a third order 3D tensor, whose mixed components are given by

$$\sigma^{ij}\|_k = \partial_k \sigma^{ij} + \Gamma^{*i}_{km}\sigma^{mj} + \Gamma^{*j}_{kn}\sigma^{in}.$$

The row divergence of the stress tensor σ^{ij} is a vector whose contravariant components are obtained from a contraction of the above third order tensor.

$$\operatorname{div}\boldsymbol{\sigma} = \sigma^{ij}\|_j = \partial_j \sigma^{ij} + \Gamma^{*i}_{jm}\sigma^{mj} + \Gamma^{*j}_{jn}\sigma^{in}. \tag{4.1.1}$$

The covariant derivative of a vector $\boldsymbol{v} = v_i$ is a second order tensor with covariant components

$$v_{i\|j} = \partial_j v_i - \Gamma^{*k}_{ij} v_k.$$

In terms of the contravariant components v^i, the mixed components of the covariant derivative of $\boldsymbol{v}$ can be expressed as

$$v^i\|_j = \partial_j v^i + \Gamma^{*i}_{kj} v^k.$$

Note that for any vector field or tensor field defined on the shell Ω^ϵ, its components can be viewed as functions defined on the coordinate domain ω^ϵ. Sometimes, we may slightly abuse notations by discarding the difference between functions defined on Ω^ϵ and ω^ϵ. The distinction should be clear from the context.

On the middle surface S, we can define the covariant and contravariant derivatives of any 2D vectors or tensors. The derivative will be denoted by a single vertical bar. A 2D tensor can be viewed as the restriction on the middle surface of a 3D tensor with zero non-tangential components. On the middle surface, the tangential part of the derivative of this 3D tensor is defined as the derivative of the 2D tensor. For example, on the surface S, the covariant derivative of the second order tensor $\underset{\approx}{\sigma} = \sigma^{\alpha\beta}$ is defined in terms of its mixed components by the first equation in (4.1.2) below. The covariant derivative of the second order tensor $\underset{\approx}{\tau} = \tau^{\alpha}_{\beta}$ is given by the second equation. The covariant derivative of the vector field $\underset{\sim}{u} = u^\alpha \boldsymbol{a}_\alpha = u_\beta \boldsymbol{a}^\beta$ is given in terms of its covariant components and mixed components by the last two equations respectively.

$$\sigma^{\alpha\beta}|_\gamma = \partial_\gamma \sigma^{\alpha\beta} + \Gamma^\alpha_{\gamma\lambda}\sigma^{\lambda\beta} + \Gamma^\beta_{\gamma\tau}\sigma^{\alpha\tau},$$

$$\tau^\gamma_{\alpha|\beta} = \partial_\beta \tau^\gamma_\alpha + \Gamma^\gamma_{\lambda\beta}\tau^\lambda_\alpha - \Gamma^\tau_{\alpha\beta}\tau^\gamma_\tau, \tag{4.1.2}$$

$$u_{\alpha|\beta} = \partial_\beta u_\alpha - \Gamma^\gamma_{\alpha\beta}u_\gamma, \quad u^\alpha|_\beta = \partial_\beta u^\alpha + \Gamma^\alpha_{\gamma\beta}u^\gamma.$$

The mixed components of the covariant derivative of the curvature tensor $b^\gamma_{\alpha|\beta} = \partial_\beta b^\gamma_\alpha + \Gamma^\gamma_{\lambda\beta}b^\lambda_\alpha - \Gamma^\tau_{\alpha\beta}b^\gamma_\tau$ is symmetric about the subscripts, i.e., $b^\gamma_{\alpha|\beta} = b^\gamma_{\beta|\alpha}$. This is the Codazzi–Mainardi identity, which follows from the second equation in (4.1.6) below. It actually is a consequence of the assumption that the surface S can be embedded in the Euclidean 3 space.

We formally define the surface covariant derivatives for the tangential parts of 3D tensors defined on the shell Ω^ϵ, by the same formulae (4.1.2), and denote them by the same notations. For example, if $\boldsymbol{\tau} = \tau^{ij}(\underset{\sim}{x}, t)$ is a tensor field defined on the shell Ω^ϵ, for any given $t_0 \in [-\epsilon, \epsilon]$, $\tau^{\alpha\beta}(\underset{\sim}{x}, t_0)$ can be viewed as the contravariant components of a 2D tensor defined on the middle surface. We will define $\tau^{\alpha\beta}|_\gamma$ at any point $(\underset{\sim}{x}, t_0)$ by the formula

$$\tau^{\alpha\beta}|_\gamma = \partial_\gamma \tau^{\alpha\beta} + \Gamma^\alpha_{\gamma\lambda}\tau^{\lambda\beta} + \Gamma^\beta_{\gamma\lambda}\tau^{\alpha\lambda}.$$

It is important to note that the derivatives denoted by a single vertical bar are always taken with respect to the metric on the middle surface. More specifically, the Christoffel symbols in the right-hand side of the above equation are those defined on the middle surface.

Product rules for differentiations like

$$(\sigma^{ij}u_j)\|_k = \sigma^{ij}\|_k u_j + \sigma^{ij}u_{j\|k},$$
$$(\sigma^{\alpha\lambda}u_\lambda)|_\beta = \sigma^{\alpha\lambda}|_\beta u_\lambda + \sigma^{\alpha\lambda}u_{\lambda|\beta}, \tag{4.1.3}$$

are, of course, always valid.

The following Green's theorem, or divergence theorem, on the surface S will be frequently used. Let $\boldsymbol{n} = n_\alpha \boldsymbol{a}^\alpha$ be the unit outward normal in the surface S to its boundary γ, then

$$\int_S u^\alpha|_\alpha dS = \int_\gamma u^\alpha n_\alpha d\gamma \tag{4.1.4}$$

holds for any vector field $\boldsymbol{u} = u^\alpha \boldsymbol{a}_\alpha$ defined on S. In the above equation, the left hand side integral is taken with respect to the surface area element and the right hand side integral is taken with respect to the arc length of the boundary curve γ.

Our ultimate goal is to approximate the 3D problem defined on the shell Ω^ϵ by a 2D problem defined on the middle surface S, so it is indispensable to make the dependence of various quantities on the transverse coordinate t as explicit as possible.

We set $\mu^\alpha_\beta(\underset{\sim}{x}, t) = \delta^\alpha_\beta - t b^\alpha_\beta(\underset{\sim}{x})$. The dependence of this tensor valued function, and of all the functions that will be introduced later, on the coordinates $(\underset{\sim}{x}, t)$ will not be indicated explicitly in the following, but should be clear from the context. We denote the determinant of μ^α_β by $\rho = \det(\mu^\alpha_\beta) = 1 - 2Ht + Kt^2$.

Let $a = \det(a_{\alpha\beta})$. Then the area element on S is $\sqrt{a}d\underset{\sim}{x}$. The volume element in the shell Ω^ϵ is $\sqrt{g}d\underset{\sim}{x}dt$. The relation $\sqrt{g} = \rho\sqrt{a}$ holds.

The mixed tensor ζ^α_β is defined as the inverse of μ^α_β (as a matrix), so $\zeta^\alpha_\gamma \mu^\gamma_\beta = \delta^\alpha_\beta$. From Cramer's rule, we have the expression

$$\zeta^\alpha_\beta = \frac{1}{\rho}\delta^{\alpha\gamma}_{\beta\lambda}\mu^\lambda_\gamma.$$

Here $\delta^{\alpha\gamma}_{\beta\lambda} = \epsilon^{\alpha\gamma}\,\epsilon_{\beta\lambda}$ is the generalized Kronecker delta. The ϵ-systems on the surface S are defined by $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\,\epsilon_{21} = \sqrt{a}$, and $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\,\epsilon^{21} = 1/\sqrt{a}$. We define the mixed tensor $d^\alpha_\beta = \delta^{\alpha\gamma}_{\beta\lambda} b^\lambda_\gamma$, which is the cofactor of the mixed curvature tensor. Then we have $\rho\zeta^\alpha_\beta = \delta^\alpha_\beta - t d^\alpha_\beta$. Note that $\rho\zeta^\alpha_\beta$ is a linear function in the transverse coordinate t. This simple observation will play an important role in our model derivation for general shells.

Between the curvature tensor b^α_β and the tensor d^α_β, the following relations hold:

$$d^\alpha_\lambda b^\lambda_\beta = K\delta^\alpha_\beta, \quad d^\alpha_\beta + b^\alpha_\beta = 2H\delta^\alpha_\beta.$$

The basis vectors and metric tensor at any point in the 3D shell are related to corresponding quantities at the projected point on the middle surface by the following equations:

$$\begin{gathered} \boldsymbol{g}_\alpha = \mu^\gamma_\alpha \boldsymbol{a}_\gamma, \quad \boldsymbol{g}^\alpha = \zeta^\alpha_\gamma \boldsymbol{a}^\gamma, \quad \boldsymbol{g}_3 = \boldsymbol{g}^3 = \boldsymbol{a}_3 = \boldsymbol{a}^3, \\ g_{\alpha\beta} = \mu^\gamma_\alpha \mu^\lambda_\beta a_{\gamma\lambda} = a_{\alpha\beta} - 2tb_{\alpha\beta} + t^2 c_{\alpha\beta},\ g_{\alpha 3} = g_{3\alpha} = 0,\ g_{33} = 1. \end{gathered} \tag{4.1.5}$$

Some important relations for the Christoffel symbols are

$$\Gamma^{*\gamma}_{33} = \Gamma^{*3}_{3\alpha} = \Gamma^{*3}_{\alpha 3} = 0,$$

$$\Gamma^{*\gamma}_{\alpha\beta} = \Gamma^{*\gamma}_{\beta\alpha} = \Gamma^{\gamma}_{\alpha\beta} - t\zeta^{\gamma}_{\lambda} b^{\lambda}_{\alpha|\beta}, \tag{4.1.6}$$

$$\Gamma^{*3}_{\alpha\beta} = b_{\alpha\beta} - tc_{\alpha\beta}, \quad \Gamma^{*\alpha}_{3\beta} = -\zeta^{\alpha}_{\lambda} b^{\lambda}_{\beta},$$

especially,

$$\Gamma^{*3}_{\alpha\beta}|_{t=0} = b_{\alpha\beta}, \quad \Gamma^{*\alpha}_{3\beta}|_{t=0} = -b^{\alpha}_{\beta}.$$

The proofs of these relations are direct applications of the definition of the Christoffel symbols. We just prove the second equation which we have not found in the literature, but is necessary for us. By the definitions of Christoffel symbols on both the middle surface S and the 3D shell Ω^{ϵ}, and the relations (4.1.5), we have

$$\begin{aligned}
\Gamma^{*\gamma}_{\alpha\beta} &= \boldsymbol{g}^{\gamma} \cdot \partial_{\beta} \boldsymbol{g}_{\alpha} = \zeta^{\gamma}_{\tau} \boldsymbol{a}^{\tau} \cdot \partial_{\beta}(\mu^{\lambda}_{\alpha} \boldsymbol{a}_{\lambda}) = \zeta^{\gamma}_{\tau} \boldsymbol{a}^{\tau} \cdot (\partial_{\beta}\mu^{\lambda}_{\alpha} \boldsymbol{a}_{\lambda} + \mu^{\lambda}_{\alpha} \partial_{\beta} \boldsymbol{a}_{\lambda}) \\
&= \zeta^{\gamma}_{\lambda}(\partial_{\beta}\mu^{\lambda}_{\alpha} + \Gamma^{\lambda}_{\tau\beta}\mu^{\tau}_{\alpha}) = \zeta^{\gamma}_{\lambda}(\mu^{\lambda}_{\alpha|\beta} + \Gamma^{\sigma}_{\alpha\beta}\mu^{\lambda}_{\sigma}) \\
&= \Gamma^{\gamma}_{\alpha\beta} + \zeta^{\gamma}_{\lambda}\mu^{\lambda}_{\alpha|\beta} = \Gamma^{\gamma}_{\alpha\beta} - t\zeta^{\gamma}_{\lambda} b^{\lambda}_{\alpha|\beta}.
\end{aligned}$$

□

Let the boundary of ω be divided to distinct parts as $\partial\omega = \partial_D\omega \cup \partial_T\omega$, with $\partial_D\omega \cap \partial_T\omega = \emptyset$, giving the clamping and traction parts of the shell lateral surface. The boundary of the middle surface S will be correspondingly divided as $\gamma = \gamma_T \cup \gamma_D$. The boundary of the shell Ω^{ϵ} is composed of the upper and lower surfaces $\Gamma_{\pm} = \boldsymbol{\Phi}(\omega \times \{\pm\,\epsilon\})$

where the shell is subjected to surface tractions, the clamping lateral surface $\Gamma_D = \mathbf{\Phi}(\partial_D\omega \times [-\epsilon, \epsilon])$ where the shell is clamped (the cross hatched part of the lateral surface in Figure 4.1), and the remaining part of the lateral surface $\Gamma_T = \mathbf{\Phi}(\partial_T\omega \times [-\epsilon, \epsilon])$, where the shell is under traction or free.

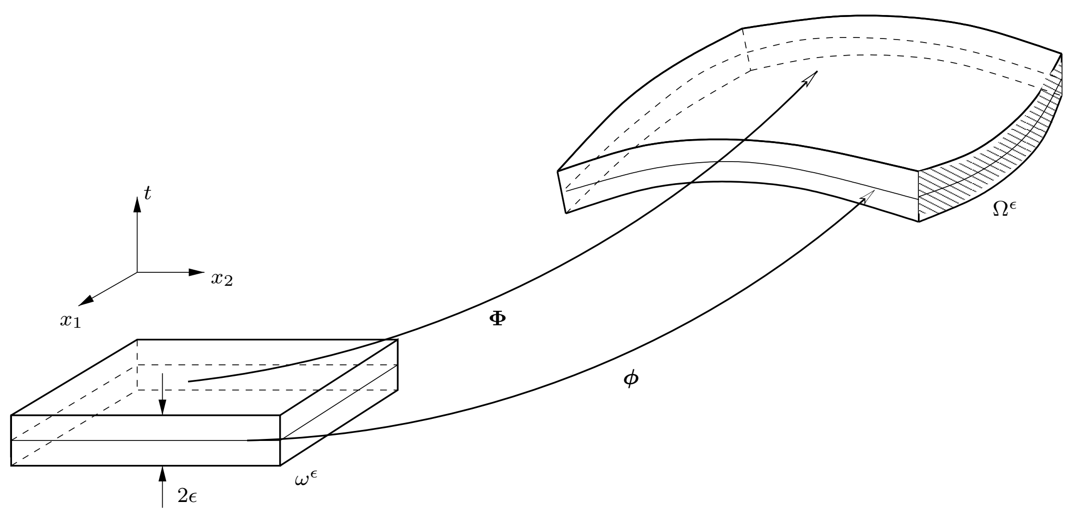


Fig. 4.1. A shell and its coordinate domain

The unit outer normal on Γ_+ is given by $\boldsymbol{n}_+ = \boldsymbol{g}^3$. The unit outer normal on Γ_- is $\boldsymbol{n}_- = -\boldsymbol{g}^3$. On Γ_T, at the point $\mathbf{\Phi}(\underset{\sim}{x}, t) \in \Gamma_T$ ($\underset{\sim}{x} \in \partial_T\omega$), we denote the unit outer normal by $\boldsymbol{n}^*$ which is obviously parallel to the middle surface, so it can be expressed as $\boldsymbol{n}^* = n^*_\alpha \boldsymbol{g}^\alpha$. Here the superscript $*$ was added to indicate the dependence of the components on t. Let $\boldsymbol{n} = n_\alpha \boldsymbol{a}^\alpha$ be the unit outer normal in the surface S to its boundary γ_T. And let $\underset{\sim}{x}(s)$ be the arc length parameterization of $\partial_T\omega$, and $\dot{\underset{\sim}{x}}(s)$ the unit tangent vector to the curve $\partial_T\omega$ at the point $\underset{\sim}{x}(s)$, then it can be shown that

$$n^*_\alpha = -\frac{\rho}{\eta}\,\epsilon_{\alpha\beta}\,\dot{x}^\beta(s).$$

As ρ measures the transverse volume variation of the shell body, the function $\eta(\underset{\sim}{x}, t)$ measures the transverse area variation of the shell lateral surface. It is given by

$$\eta(\underset{\sim}{x}(s), t) = \frac{\sqrt{g_{\alpha\beta}(\underset{\sim}{x}(s), t)\dot{x}^\alpha(s)\dot{x}^\beta(s)}}{\sqrt{a_{\alpha\beta}(\underset{\sim}{x}(s))\dot{x}^\alpha(s)\dot{x}^\beta(s)}} \quad \forall\ \underset{\sim}{x}(s) \in \partial_T\omega,$$

so we have

$$n^*_\alpha = \frac{\rho}{\eta} n_\alpha. \tag{4.1.7}$$

4.2 Linearized elasticity theory

In the context of the linearized elasticity, the deformation and stress distribution in an elastic shell arising in response to the applied forces and boundary conditions are determined by the geometric equation (4.2.1), the constitutive equation (4.2.2), the equilibrium equation (4.2.3), and traction (4.2.4) and clamping (4.2.5) boundary conditions on the shell surface.

Let the surface force densities on $\Gamma_\pm$ be $\boldsymbol{p}_\pm = p^i_\pm \boldsymbol{g}_i$, the surface force density on Γ_T be $\boldsymbol{p}_T = p^i_T \boldsymbol{g}_i$, and the body force density be $\boldsymbol{q} = q^i \boldsymbol{g}_i$. Note that $\boldsymbol{g}_i$ are the covariant basis vectors at the relevant point.

Let $\boldsymbol{v}$ be the displacement vector field, χ_{ij} the strain tensor field, and σ^{ij} the stress tensor field. The displacement-strain relation, or geometric equation, is

$$\chi_{ij} = \frac{1}{2}(v_{i\|j} + v_{j\|i}). \tag{4.2.1}$$

The constitutive equation, which connects stress to strain, is

$$\sigma^{ij} = C^{ijkl}\chi_{kl} \quad \text{or} \quad \chi_{ij} = A_{ijkl}\sigma^{kl}, \tag{4.2.2}$$

where the 3D fourth order tensors C^{ijkl} and A_{ijkl} are the elasticity tensor and the compliance tensor. They are given by

$$C^{ijkl} = 2\mu g^{ik}g^{jl} + \lambda g^{ij}g^{kl} \quad \text{and} \quad A_{ijkl} = \frac{1}{2\mu}[g_{ik}g_{jl} - \frac{\lambda}{2\mu + 3\lambda}g_{ij}g_{kl}],$$

respectively. The equilibrium equation, expressed in terms of the tensor and vector components, is

$$\sigma^{ij}\|_j + q^i = 0. \tag{4.2.3}$$

On $\Gamma_\pm$ and Γ_T, the surface force condition, expressed in terms of the contravariant stress components, is

$$\sigma^{3j} = p_+^j \text{ on } \Gamma_+; \quad \sigma^{3j} = -p_-^j \text{ on } \Gamma_-; \quad \sigma^{j\alpha}n_\alpha^* = p_T^j \text{ on } \Gamma_T. \tag{4.2.4}$$

On Γ_D the shell is clamped, so the displacement vanishes, and the condition is

$$v_i = 0 \text{ on } \Gamma_D. \tag{4.2.5}$$

The theory of linearized 3D elasticity says that the system of equations (4.2.1), (4.2.2), (4.2.3), together with the boundary conditions (4.2.4) and (4.2.5) uniquely determine the displacement $\boldsymbol{v}^* = v_i^*$ and the stress $\boldsymbol{\sigma}^* = \sigma^{*ij}$ distributions over the loaded shell arising in response to the applied body force, surface force, and clamping boundary condition. The displacement $\boldsymbol{v}^*$ can be determined as the unique solution of the weak form of the 3D elasticity equations:

$$\begin{gathered} \int_{\Omega^\epsilon} C^{ijkl}\chi_{kl}(\boldsymbol{v})\chi_{ij}(\boldsymbol{u}) = \int_{\Omega^\epsilon} q^i u_i + \int_{\Gamma_\pm} p_\pm^i u_i + \int_{\Gamma_T} p_T^i u_i, \\ \boldsymbol{v} \in \boldsymbol{H}_D^1(\omega^\epsilon), \quad \forall \boldsymbol{u} \in \boldsymbol{H}_D^1(\omega^\epsilon). \end{gathered} \tag{4.2.6}$$

where $\boldsymbol{H}_D^1(\omega^\epsilon)$ is the space of vector valued functions whose components and first derivatives are square integrable on ω^ϵ, and whose value vanish on Γ_D. For any given body force density $\boldsymbol{q} = q^i \boldsymbol{g}_i$ with q^i in the dual space of $\boldsymbol{H}_D^1(\omega^\epsilon)$ and traction surface force densities $\boldsymbol{p}_\pm$ and $\boldsymbol{p}_T$ with contravariant components $p_\pm^i$ and p_T^i together defining a functional on $\boldsymbol{H}_{00}^{1/2}(\Gamma_\pm \cup \Gamma_T)$, this variational problem uniquely determine the displacement vector field $\boldsymbol{v}^* = v_i^* \in \boldsymbol{H}_D^1(\omega^\epsilon)$. With the unique displacement solution $\boldsymbol{v}^*$ of the 3D elasticity equations determined, through the geometric equation (4.2.1) and the constitutive equation (4.2.2), we can determine the stress tensor $\boldsymbol{\sigma}^* = \sigma^{*ij}$.

A stress field $\boldsymbol{\sigma} = \sigma^{ij}$ is said to be statically admissible, if it satisfies the equilibrium equation (4.2.3) and the traction boundary condition (4.2.4). A displacement field $\boldsymbol{v} = v_i \in \boldsymbol{H}^1(\Omega)$ is kinematically admissible, if it satisfies the clamping boundary condition (4.2.5). If both $\boldsymbol{\sigma}$ and $\boldsymbol{v}$ are admissible, the following identity holds:

$$\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)]$$
$$= \int_{\Omega^\epsilon} [\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})][A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})]. \quad (4.2.7)$$

This is the two energies principle. For spherical shells, our model derivation and justification are based on this identity.

4.3 Rescaled components

Due to the complicated expression (4.1.1), it is quite difficult to compute the row divergence of a stress tensor given by its contravariant components. We will need to verify the admissibility of a stress field in the justification of the spherical shell model, and need to compute the residual of the equilibrium equation of a stress field for the justification of the general shell model. So the calculation of the row divergence of stress field is absolutely necessary. To simplify the calculation, we introduce the rescaled stress components $\tilde{\sigma}^{ij}$ for a stress field σ^{ij} by defining

$$\tilde{\sigma}^{\alpha\beta} = \rho\mu^\alpha_\gamma\sigma^{\gamma\beta}, \quad \tilde{\sigma}^{3\alpha} = \rho\sigma^{3\alpha}, \quad \tilde{\sigma}^{\alpha 3} = \rho\sigma^{\alpha 3}, \quad \tilde{\sigma}^{33} = \rho\sigma^{33}, \quad (4.3.1)$$

or equivalently

$$\sigma^{\alpha\beta} = \frac{1}{\rho}\zeta^{\alpha}_{\gamma}\tilde{\sigma}^{\gamma\beta}, \quad \sigma^{3\alpha} = \frac{1}{\rho}\tilde{\sigma}^{3\alpha}, \quad \sigma^{\alpha 3} = \frac{1}{\rho}\tilde{\sigma}^{\alpha 3}, \quad \sigma^{33} = \frac{1}{\rho}\tilde{\sigma}^{33}. \tag{4.3.2}$$

The following lemma indicates that, in terms of the rescaled stress components, the divergence of the stress tensor σ^{ij} has a simpler form.

LEMMA 4.3.1. *In terms of the rescaled components $\tilde{\sigma}^{ij}$, the row divergence of the stress tensor σ^{ij} has the expression*

$$\begin{aligned} \sigma^{\alpha j}\|_j &= \frac{1}{\rho}\zeta^{\alpha}_{\gamma}[\tilde{\sigma}^{\gamma\beta}|_{\beta} + \mu^{\gamma}_{\lambda}\partial_t\tilde{\sigma}^{\lambda 3} - 2b^{\gamma}_{\tau}\tilde{\sigma}^{\tau 3}], \\ \sigma^{3j}\|_j &= \frac{1}{\rho}[\tilde{\sigma}^{3\beta}|_{\beta} + \partial_t\tilde{\sigma}^{33} + b_{\gamma\lambda}\tilde{\sigma}^{\gamma\lambda}]. \end{aligned} \tag{4.3.3}$$

Note that, the derivatives in the right hand side are all taken with respect to the metric of the middle surface of the shell.

Proof. For the first equation, on one hand, by the relations (4.1.6), we have

$$\sigma^{\alpha j}\|_j = \partial_j\sigma^{\alpha j} + \Gamma^{*\gamma}_{i\gamma}\sigma^{\alpha i} + \Gamma^{*\alpha}_{ij}\sigma^{ij} = \partial_j\sigma^{\alpha j} + \Gamma^{*\gamma}_{\lambda\gamma}\sigma^{\alpha\lambda} + \Gamma^{*\gamma}_{3\gamma}\sigma^{\alpha 3} + \Gamma^{*\alpha}_{\delta\tau}\sigma^{\delta\tau} + 2\Gamma^{*\alpha}_{3\tau}\sigma^{3\tau}.$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\rho}\zeta^{\alpha}_{\gamma}[\ \tilde{\sigma}\ ^{\gamma\beta}|_{\beta} + \mu^{\gamma}_{\lambda}\partial_t\tilde{\sigma}^{\lambda 3} - 2b^{\gamma}_{\tau}\tilde{\sigma}^{\tau 3}] \\ &\quad = \frac{1}{\rho}\zeta^{\alpha}_{\gamma}[(\rho\mu^{\gamma}_{\tau}\sigma^{\tau\beta})|_{\beta} + \mu^{\gamma}_{\lambda}\partial_t(\rho\sigma^{\lambda 3}) - 2b^{\gamma}_{\tau}\rho\sigma^{\tau 3}] \\ &\quad = \frac{1}{\rho}\zeta^{\alpha}_{\gamma}[\partial_{\beta}\rho\mu^{\gamma}_{\tau}\sigma^{\tau\beta} + \rho\mu^{\gamma}_{\tau|\beta}\sigma^{\tau\beta} + \rho\mu^{\gamma}_{\tau}\sigma^{\tau\beta}|_{\beta} + \mu^{\gamma}_{\lambda}\partial_t\rho\sigma^{\lambda 3} + \rho\partial_t\sigma^{\lambda 3} - 2b^{\gamma}_{\tau}\rho\sigma^{\tau 3}] \end{aligned}$$

$$= \frac{1}{\rho}\partial_\beta \rho \sigma^{\alpha\beta} + \zeta^\alpha_\lambda \mu^\lambda_{\tau|\beta}\sigma^{\tau\beta} + \sigma^{\alpha\beta}|_\beta + \frac{1}{\rho}\partial_t \rho \sigma^{\alpha 3} + \partial_t \sigma^{\alpha 3} - 2\zeta^\alpha_\lambda b^\lambda_\beta \sigma^{\beta 3}$$

$$= (\Gamma^{*\gamma}_{\gamma\beta} - \Gamma^\gamma_{\gamma\beta})\sigma^{\alpha\beta} + (\Gamma^{*\alpha}_{\gamma\beta} - \Gamma^\alpha_{\gamma\beta})\sigma^{\gamma\beta} + \partial_\beta \sigma^{\alpha\beta}$$

$$+ \Gamma^\beta_{\gamma\beta}\sigma^{\alpha\gamma} + \Gamma^\alpha_{\lambda\beta}\sigma^{\lambda\beta} + \Gamma^{*\gamma}_{\gamma 3}\sigma^{\alpha 3} + \partial_t \sigma^{\alpha 3} + 2\Gamma^{*\alpha}_{\beta 3}\sigma^{\beta 3}$$

$$= \partial_\beta \sigma^{\alpha\beta} + \partial_t \sigma^{\alpha 3} + \Gamma^{*\gamma}_{\gamma\beta}\sigma^{\alpha\beta} + \Gamma^{*\alpha}_{\gamma\beta}\sigma^{\gamma\beta} + \Gamma^{*\gamma}_{\gamma 3}\sigma^{\alpha 3} + 2\Gamma^{*\alpha}_{\beta 3}\sigma^{\beta 3}.$$

The first equation in (4.3.3) then follows. In the above calculation, the following identities were used [30].

$$\Gamma^{*i}_{ij} = \frac{\partial_j \sqrt{g}}{\sqrt{g}}, \quad \frac{1}{\rho}\partial_\beta \rho = \frac{\partial_\beta \sqrt{g}}{\sqrt{g}} - \frac{\partial_\beta \sqrt{a}}{\sqrt{a}} = \Gamma^{*\gamma}_{\gamma\beta} - \Gamma^\gamma_{\gamma\beta}, \quad \frac{1}{\rho}\partial_t \rho = \frac{\partial_t \sqrt{g}}{\sqrt{g}} = \Gamma^{*\gamma}_{\gamma 3}.$$

For the second equation, we have

$$\sigma^{3j}\|_j = \partial_j \sigma^{3j} + \Gamma^{*3}_{\alpha\beta}\sigma^{\alpha\beta} + \Gamma^{*\lambda}_{\lambda n}\sigma^{3n},$$

and

$$\frac{1}{\rho}[\ \tilde{\sigma}\ ^{3\beta}|_\beta + \partial_t \tilde{\sigma}^{33} + b_{\gamma\lambda}\tilde{\sigma}^{\gamma\lambda}]$$

$$= \frac{1}{\rho}[(\rho\sigma^{3\beta})|_\beta + \partial_t(\rho\sigma^{33}) + b_{\gamma\lambda}\rho\mu^\gamma_\tau \sigma^{\tau\lambda}]$$

$$= \sigma^{3\beta}|_\beta + \frac{\partial_\beta \rho}{\rho}\sigma^{3\beta} + \partial_t \sigma^{33} + \frac{\partial_t \rho}{\rho}\sigma^{33} + b_{\gamma\lambda}\mu^\gamma_\tau \sigma^{\tau\lambda}$$

$$= \partial_\beta \sigma^{3\beta} + \Gamma^\beta_{\gamma\beta}\sigma^{3\gamma} + (\Gamma^{*\gamma}_{\gamma\beta} - \Gamma^\gamma_{\gamma\beta})\sigma^{3\beta} + \partial_t \sigma^{33} + \Gamma^{*\gamma}_{\gamma 3}\sigma^{33} + \Gamma^{*3}_{\tau\lambda}\sigma^{\tau\lambda}$$

$$= \partial_j \sigma^{3j} + \Gamma^{*3}_{\alpha\beta}\sigma^{\alpha\beta} + \Gamma^{*\lambda}_{\lambda n}\sigma^{3n}.$$

The desired equation follows. □

Note that except for some special shells, for example, plates and spherical shells, the rescaled stress components $\tilde{\sigma}^{ij}$ is not symmetric, more specifically, $\tilde{\sigma}^{12} \neq \tilde{\sigma}^{21}$.

For consistency with the rescaled stress components, we introduce rescaled components for the applied forces. For the upper and lower surface force densities $\boldsymbol{p}_{\pm}$, we introduce the rescaled components $\tilde{p}^i_{\pm}$ by the relation

$$\boldsymbol{p}_{\pm} = p^i_{\pm}\boldsymbol{g}_i = \tilde{p}^i_{\pm}\frac{1}{\rho}\boldsymbol{g}_i, \tag{4.3.4}$$

where $p^i_{\pm}$ are the usual contravariant components of the surface forces. The rescaled components $\tilde{p}^i_{\pm}$ take the diffences of the areas of the upper and lower surfaces from that of the middle surface into account.

For the lateral surface force density $\boldsymbol{p}_T$, we introduce the rescaled components $\tilde{p}^i_T$ by the relation

$$\boldsymbol{p}_T = p^i_T\boldsymbol{g}_i = \tilde{p}^i_T\frac{1}{\eta}\boldsymbol{a}_i. \tag{4.3.5}$$

The rescaled components account the transverse area variation of the lateral surface, and more explicitly express the dependence of the lateral surface force density on t.

For the body force density $\boldsymbol{q}$, we define the new components $\tilde{q}^i$ by

$$\boldsymbol{q} = q^i\boldsymbol{g}_i = \tilde{q}^i\frac{1}{\rho}\boldsymbol{a}_i, \tag{4.3.6}$$

where q^i are the contravariant components of the body force density, while $\tilde{q}^i$ are the components of the body force density weighted by the transverse volume change, and expressed in terms of the covariant basis on the middle surface.

In terms of the rescaled stress components and applied forces components, the surface force condition (4.2.4) can be equivalently written as

$$\tilde{\sigma}^{3j} = \tilde{p}_+^j \text{ on } \Gamma_+; \quad \tilde{\sigma}^{3j} = -\tilde{p}_-^j \text{ on } \Gamma_-; \quad \tilde{\sigma}^{j\alpha} n_\alpha = \tilde{p}_T^j \text{ on } \Gamma_T. \tag{4.3.7}$$

The equilibrium residual $\sigma^{ij}\|_j + q^i$ can be equally written as

$$\begin{aligned} \sigma^{\alpha j}\|j + q^\alpha &= \frac{1}{\rho}\zeta_\gamma^\alpha[\tilde{\sigma}^{\gamma\beta}|_\beta + \mu_\lambda^\gamma \partial_t \tilde{\sigma}^{\lambda 3} - 2b_\tau^\gamma \tilde{\sigma}^{\tau 3} + \tilde{q}^\gamma], \\ \sigma^{3j}\|_j + q^3 &= \frac{1}{\rho}[\tilde{\sigma}^{3\beta}|_\beta + \partial_t \tilde{\sigma}^{33} + b_{\gamma\lambda}\tilde{\sigma}^{\gamma\lambda} + \tilde{q}^3]. \end{aligned} \tag{4.3.8}$$

For the displacement vector $\boldsymbol{v} = v_i \boldsymbol{g}^i$, we introduce the rescaled components $\tilde{v}_i$ by expressing the vector in terms of the basis vectors on the middle surface, i.e., $\boldsymbol{v} = \tilde{v}_i \boldsymbol{a}^i$. In components, the relation is

$$v_\alpha(x_1, x_2, t) = \mu_\alpha^\gamma \tilde{v}_\gamma(x_1, x_2, t), \quad v_3(x_1, x_2, t) = \tilde{v}_3(x_1, x_2, t).$$

Lemma 4.3.2. *In terms of the rescaled components $\tilde{v}_i$ of the displacement vector field $\boldsymbol{v}$, the strain tensor engendered by $\boldsymbol{v}$ can be expressed as*

$$\chi_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(\tilde{v}_{\alpha|\beta} + \tilde{v}_{\beta|\alpha} - 2b_{\alpha\beta}\tilde{v}_3) - \frac{1}{2}t(b_\alpha^\gamma \tilde{v}_{\gamma|\beta} + b_\beta^\gamma \tilde{v}_{\gamma|\alpha} - 2c_{\alpha\beta}\tilde{v}_3),$$

$$\chi_{\alpha 3}(\boldsymbol{v}) = \chi_{3\alpha}(\boldsymbol{v}) = \frac{1}{2}(\partial_\alpha \tilde{v}_3 + \partial_t \tilde{v}_\alpha + b_\alpha^\gamma \tilde{v}_\gamma - t b_\alpha^\gamma \partial_t \tilde{v}_\gamma), \quad \chi_{33}(\boldsymbol{v}) = \partial_t \tilde{v}_3.$$

Proof. We need to compute the covariant derivatives $v_{i\|j} = \partial_j \boldsymbol{v} \cdot \boldsymbol{g}_i$. By direct computation, we see

$$\partial_\beta \boldsymbol{v} = \partial_\beta \tilde{v}_\gamma \boldsymbol{a}^\gamma + \tilde{v}_\gamma \partial_\beta \boldsymbol{a}^\gamma + \partial_\beta \tilde{v}_3 \boldsymbol{a}^3 + \tilde{v}_3 \partial_\beta \boldsymbol{a}^3.$$

Using the definitions of the Christoffel symbols, curvature tensors, and covariant derivatives on the middle surface to the right hand sides of this equation, we get

$$\partial_\beta \boldsymbol{v} = (\tilde{v}_{\gamma|\beta} - b_{\gamma\beta}\tilde{v}_3)\boldsymbol{a}^\gamma + (\partial_\beta \tilde{v}_3 + b_\beta^\gamma \tilde{v}_\gamma)\boldsymbol{a}^3.$$

Therefore,

$$v_{\alpha\|\beta} = \partial_\beta \boldsymbol{v} \cdot \mu_\alpha^\lambda \boldsymbol{a}_\lambda = \mu_\alpha^\lambda(\tilde{v}_{\lambda|\beta} - b_{\lambda\beta}\tilde{v}_3) = \tilde{v}_{\alpha|\beta} - b_{\alpha\beta}\tilde{v}_3 - t b_\alpha^\gamma \tilde{v}_{\gamma|\beta} + t c_{\alpha\beta}\tilde{v}_3,$$

$$v_{3\|\beta} = \partial_\beta \boldsymbol{v} \cdot \boldsymbol{g}_3 = \partial_\beta \tilde{v}_3 + b_\beta^\gamma \tilde{v}_\gamma.$$

It is easy to see that $\partial_3 \boldsymbol{v} = \partial_t \tilde{v}_\gamma \boldsymbol{a}^\gamma + \partial_t \tilde{v}_3 \boldsymbol{a}^3$, so, we have $v_{\beta\|3} = \partial_3 \boldsymbol{v} \cdot \boldsymbol{g}_\beta = \mu_\beta^\gamma \partial_t \tilde{v}_\gamma = \partial_t \tilde{v}_\beta - t b_\beta^\gamma \partial_t \tilde{v}_\gamma$ and $v_{3\|3} = \partial_t \tilde{v}_3$. The lemma then follows from the definition of the strain tensor (4.2.1). $\square$

Chapter 5

Spherical shell model

5.1 Introduction

In this chapter, we discuss the 2D modeling of the deformation of a thin shell whose middle surface is a portion of a sphere. The shell can be totally or partially clamped. The model is constructed in the vein of the minimum complementary energy principle, and will be justified by the two energies principle. The form of the model is similar to that of the plane strain cylindrical shells justified in Chapter 2, and can be put in the abstract framework of Chapter 3. Since the membrane–shear operator B does not have closed range, the behavior of the model solution is more complicated, and the justification is more difficult. For totally clamped spherical shells, convergence in the relative energy norm of the 2D model solution to the 3D elasticity solution is proved. A convergence rate of $O(\epsilon^{1/6})$ in the relative energy norm is established under some smoothness assumption on the shell data in the usual Sobolev sense. For partially clamped spherical shells, convergence and convergence rate will be proved under a condition imposed on an ϵ-independent 2D problem. This condition is an indirect requirement on the regularity of the shell data, whose interpretation in the usual Sobolev sense is not completely clear yet. An example for which the shell model might not be applicable will be given.

The spherical shell problem is another example that can be resolved by the two energies principle. Together with the plane strain cylindrical shells, these special shell problems provide examples for all kinds of shells as classified in the next chapter.

5.2 Three-dimensional spherical shells

A spherical shell is a special shell, to which all the definitions and equations of Chapter 4 apply. Here, we summarize the things that are special to spherical shells.

The middle surface S of the spherical shell is a portion of a sphere of radius R. A spherical shell, with middle surface S and thickness 2ϵ, is a 3D elastic body occupying the domain $\Omega^\epsilon \subset \mathbb{R}^3$, which is the image of a plate-like domain ω^ϵ through the mapping $\mathbf{\Phi}$ defined in Chapter 4. We assume $\epsilon < R$ so that the mapping $\mathbf{\Phi}$ is injective. Through the mapping $\mathbf{\Phi}$, the Cartesian coordinates on ω^ϵ furnish the curvilinear coordinates on the shell Ω^ϵ. The peculiarity of the spherical shell Ω^ϵ lies in the fact that the mixed curvature tensor of its middle surface is a scalar multiple of the Kronecker δ: $b^\alpha_\beta = b\delta^\alpha_\beta$, with $b = -1/R$. To see this, we introduce the spherical coordinates on the middle surface (x_1 for the longitudes and x_2 for the latitudes) and let the normal direction point outward. With these coordinates, the covariant components of metric tensor are

$$a_{11} = R^2 \cos^2 x_2,\ a_{22} = R^2,\ a_{12} = a_{21} = 0.$$

The covariant components of the curvature tensor are

$$b_{11} = -R \cos^2 x_2,\ b_{22} = -R,\ b_{12} = b_{21} = 0.$$

Note that, the mixed curvature tensor is given by $b^\alpha_\beta = b\delta^\alpha_\beta$. We know that when the curvilinear coordinates are changed, the mixed components of a second order tensor change according to the rule of similarity matrix transformation. Therefore, on a sphere, the mixed curvature tensor always takes this special form, no matter what coordinates are used. Because of the special form of the mixed curvature tensor, we have the following special relations that will substantially simplify the analysis.

$$
\begin{gathered}
\mu^\alpha_\beta = (1-bt)\delta^\alpha_\beta, \quad \zeta^\alpha_\beta = \frac{1}{1-bt}\delta^\alpha_\beta. \\
H = b, \quad K = b^2, \quad \rho = (1-bt)^2, \quad \eta = 1-bt, \\
\boldsymbol{g}_\alpha = (1-bt)\boldsymbol{a}_\alpha, \quad \boldsymbol{g}^\alpha = \frac{1}{1-bt}\boldsymbol{a}^\alpha, \\
g_{\alpha\beta} = (1-bt)^2 a_{\alpha\beta}, \quad b_{\alpha\beta} = b a_{\alpha\beta}, \quad c_{\alpha\beta} = b^2 a_{\alpha\beta}.
\end{gathered}
\tag{5.2.1}
$$

For the spherical shell, the rescaled stress components that was defined for general shells in (4.3.1) become

$$
\tilde{\sigma}^{\alpha\beta} = (1-bt)^3\sigma^{\gamma\beta}, \quad \tilde{\sigma}^{\alpha 3} = \tilde{\sigma}^{3\alpha} = (1-bt)^2\sigma^{3\alpha}, \quad \tilde{\sigma}^{33} = (1-bt)^2\sigma^{33}. \tag{5.2.2}
$$

Note that the matrix of rescaled stress components is symmetric, a property particular to spherical shells. By using the equation (4.3.3), we can write the divergence of a stress field in terms of the rescaled stress components as

$$\sigma^{\alpha j}\|_j = \frac{1}{(1-bt)^3}[\tilde{\sigma}^{\alpha\beta}|_\beta + (1-bt)\partial_t\tilde{\sigma}^{\alpha 3} - 2b\tilde{\sigma}^{\alpha 3}],$$
$$\sigma^{3j}\|_j = \frac{1}{(1-bt)^2}[\tilde{\sigma}^{3\beta}|_\beta + \partial_t\tilde{\sigma}^{33} + ba_{\gamma\lambda}\tilde{\sigma}^{\gamma\lambda}]. \tag{5.2.3}$$

The shell is subjected to surface forces $\boldsymbol{p}_\pm$ on $\Gamma_\pm$, and $\boldsymbol{p}_T$ on Γ_T per unit area. It is loaded by a body force $\boldsymbol{q}$ per unit volume. The shell is clamped on Γ_D. The rescaled components of the applied forces are connected to the contravariant components through the relations, see (4.3.4), (4.3.5), and (4.3.6),

$$\boldsymbol{p}_\pm = p^i_\pm \boldsymbol{g}_i = \tilde{p}^i_\pm \frac{1}{\rho}\boldsymbol{g}_i, \quad \boldsymbol{p}_T = p^i_T \boldsymbol{g}_i = \tilde{p}^i_T \frac{1}{\eta}\boldsymbol{a}_i, \quad \boldsymbol{q} = q^i \boldsymbol{g}_i = \tilde{q}^i \frac{1}{\rho}\boldsymbol{a}_i. \tag{5.2.4}$$

In terms of the rescaled stress components $\tilde{\sigma}^{ij}$ and the rescaled applied force components, the equilibrium equation $\sigma^{ij}\|_j + q^i = 0$ can be equivalently written as, see (4.3.8),

$$\tilde{\sigma}^{\alpha\beta}|_\beta + (1-bt)\partial_t\tilde{\sigma}^{3\alpha} - 2b\tilde{\sigma}^{3\alpha} + \tilde{q}^\alpha = 0,$$
$$\tilde{\sigma}^{3\beta}|_\beta + \partial_t\tilde{\sigma}^{33} + ba_{\gamma\lambda}\tilde{\sigma}^{\gamma\lambda} + \tilde{q}^3 = 0. \tag{5.2.5}$$

The unit outer normal vector on the upper surface Γ_+ is obviously given by $\boldsymbol{n}_+ = \boldsymbol{g}^3$ and on the lower surface Γ_-, $\boldsymbol{n}_- = -\boldsymbol{g}^3$. The surface force conditions $\sigma^{ij}n_i = p^j_\pm$ on $\Gamma_\pm$ are equivalent to

$$\tilde{\sigma}^{3\alpha}(\epsilon) = \tilde{p}^\alpha_+, \quad \tilde{\sigma}^{3\alpha}(-\epsilon) = -\tilde{p}^\alpha_-, \quad \tilde{\sigma}^{33}(\epsilon) = \tilde{p}^3_+, \quad \tilde{\sigma}^{33}(-\epsilon) = -\tilde{p}^3_-. \tag{5.2.6}$$

On the lateral surface Γ_T, let the unit outer normal vector at a point on the middle curve γ_T be $\boldsymbol{n} = n_\alpha \boldsymbol{a}^\alpha$, which should be in the middle surface. Note that along the vertical straight fiber through this point, the unit outer normal should not change, so $\boldsymbol{n}^* = n_i^* \boldsymbol{g}^i = n_\alpha \boldsymbol{a}^\alpha$. The components are $n_\alpha^* = (1 - bt) n_\alpha, \quad n_3^* = 0$. The lateral surface force condition $\sigma^{ij} n_i^* = p_T^j$ on Γ_T, can be equivalently written as

$$\tilde{\sigma}^{\alpha\beta} n_\beta = \tilde{p}_T^\alpha, \quad \tilde{\sigma}^{3\beta} n_\beta = \tilde{p}_T^3. \tag{5.2.7}$$

In terms of the rescaled applied surface force components, we define the odd and weighted even parts of the surface forces by

$$p_o^\alpha = \frac{\tilde{p}_+^\alpha - \tilde{p}_-^\alpha}{2}, \quad p_e^\alpha = \frac{\tilde{p}_+^\alpha + \tilde{p}_-^\alpha}{2\,\epsilon}, \quad p_o^3 = \frac{\tilde{p}_+^3 - \tilde{p}_-^3}{2}, \quad p_e^3 = \frac{\tilde{p}_+^3 + \tilde{p}_-^3}{2\,\epsilon}. \tag{5.2.8}$$

For the body force, we define the components of the transverse average by

$$q_a^i = \frac{1}{2\,\epsilon} \int_{-\,\epsilon}^{\epsilon} \boldsymbol{q} \cdot \boldsymbol{a}^i dt.$$

We assume that the body force density $\boldsymbol{q}$ is constant in the transverse coordinate. This is equivalent to $\boldsymbol{q} = q_a^i \boldsymbol{a}_i$. Under this assumption, the rescaled body force components are quadratic polynomials in t, and we have $\tilde{q}^i = q_0^i + t q_1^i + t^2 q_2^i$, with $q_0^i = q_a^i$, $q_1^i = -2b q_a^i$, and $q_2^i = b^2 q_a^i$.

For the lateral surface force, we define the components of the transverse average and moment by

$$p_a^i = \frac{1}{2\,\epsilon}\int_{-\,\epsilon}^{\epsilon} \boldsymbol{p}_T \cdot \boldsymbol{a}^i dt, \quad p_m^i = \frac{3}{2\,\epsilon^3}\int_{-\,\epsilon}^{\epsilon} t\boldsymbol{p}_T \cdot \boldsymbol{a}^i dt.$$

We assume that the lateral surface force density $\boldsymbol{p}_T$ changes linearly in the transverse coordinate, or equivalently $\boldsymbol{p}_T = (p_a^i + tp_m^i)\boldsymbol{a}_i$. Under this assumption, the rescaled lateral surface force components $\tilde{p}_T^i$ are quadratic functions in t, and we have $\tilde{p}_T^i = p_0^i + tp_1^i + t^2 p_2^i$, with $p_0^i = p_a^i$, $p_1^i = p_m^i - bp_a^i$, and $p_2^i = -bp_m^i$. The following analyses can be carried through if $\tilde{q}^i$ and $\tilde{p}_T^i$ are arbitrary quadratic polynomials in t. The restriction on the body force density and lateral surface force density can be further relaxed, see Remark 6.3.1.

5.3 The spherical shell model

The model is a 2D variational problem defined on the space $H = \underset{\sim}{H}{}_D^1(\omega) \times \underset{\sim}{H}{}_D^1(\omega) \times H_D^1(\omega)$. The solution of the model is composed of five two variable functions that can approximately describe the shell displacement arising in response to the applied loads and boundary conditions. For $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$, we define

$$\gamma_{\alpha\beta}(\underset{\sim}{u}, w) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - ba_{\alpha\beta}w,$$
$$\rho_{\alpha\beta}(\underset{\sim}{\theta}) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}), \quad \tau_\beta(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \theta_\beta + \partial_\beta w + bu_\beta, \tag{5.3.1}$$

which give the membrane, flexural, and transverse shear strains engendered by the displacement functions $(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$. The model reads: Find $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \in H$, such that

$$\frac{1}{3}\epsilon^2 \int_\omega a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}(\underset{\sim}{\theta}^\epsilon)\rho_{\alpha\beta}(\underset{\sim}{\phi})\sqrt{a}d\underset{\sim}{x}$$

$$+ \int_\omega a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}(\underset{\sim}{u}^\epsilon, w^\epsilon)\gamma_{\alpha\beta}(\underset{\sim}{v}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu \int_\omega a^{\alpha\beta}\tau_\beta(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{v}, z)\sqrt{a}d\underset{\sim}{x}$$

$$= \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H, \quad (5.3.2)$$

where $a^{\alpha\beta\lambda\gamma} = 2\mu a^{\alpha\lambda}a^{\beta\gamma} + \lambda^\star a^{\alpha\beta}a^{\lambda\gamma}$ is the 2D elasticity tensor of the shell and

$$\lambda^\star = \frac{2\mu\lambda}{2\mu + \lambda}.$$

The leading term in the resultant loading functional is given by

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{5}{6}\int_\omega p_o^\alpha \tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} - \frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$

$$+ \int_\omega [(q_a^\alpha - 2bp_o^\alpha + p_e^\alpha)y_\alpha + (q_a^3 + p_o^\alpha|_\alpha + p_e^3)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p_a^\alpha y_\alpha, \quad (5.3.3)$$

and the higher order term is

$$\langle \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = -\frac{\lambda}{3(2\mu+\lambda)}\int_\omega (p_e^3 + bp_o^3)a^{\alpha\beta}\rho_{\alpha\beta}(\underset{\sim}{\phi})\sqrt{a}d\underset{\sim}{x}$$

$$+ \frac{1}{3}b\int_\omega [bq_a^\alpha y_\alpha + bq_a^3 z - (3p_e^\alpha + 2q_a^\alpha)\phi_\alpha]\sqrt{a}d\underset{\sim}{x}$$

$$+ \frac{1}{3}\int_{\gamma_T} [-bp_m^\alpha y_\alpha + 2bp_m^3 z + (p_m^\alpha - bp_a^\alpha)\phi_\alpha]. \quad (5.3.4)$$

Remark 5.3.1. *It is noteworthy that the leading term p_a^3 of the transverse component of the lateral surface force is not incorporated in the expression of the leading term of the resultant loading functional $\boldsymbol{f}_0$. Our explanation for this unreasonable phenomena is that the effect of p_a^3 is represented by the odd part of the upper and lower surface forces p_o^α through the compatibility condition (5.4.9).*

This is the variational formulation of our spherical shell model. This model is a close variant of the classical Naghdi model. The differences lie in the shear correction factor 5/6, and more significantly, the expression of the flexural strain

$$\rho_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}).$$

The flexural strain in the Naghdi model is given by

$$\rho_{\alpha\beta}^N = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) - b\gamma_{\alpha\beta}(\underset{\sim}{u}, w)$$

where $\gamma_{\alpha\beta}$ is the membrane strain defined in (5.3.1).

We will derive a model for general shells in Chapter 6. When the general shell model is applied to spherical shells, a spherical shell model that is slightly different from the one we derived here will be obtained. Especially, the flexural strain will be given by

$$\rho_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + b\gamma_{\alpha\beta}(\underset{\sim}{u}, w).$$

It seems that the model (5.3.2) is closer to that of Budianski–Sanders [14].

We will prove the convergence of the spherical shell model in the next section, and prove the convergence for general shell model in the next chapter. The discrepancy can be explained by the difference in the resultant loading functional. What we can learn from the difference between the two spherical shell models we derived is that the model can be changed, but the crux is that the resultant loading functional must be changed accordingly, otherwise, a variant in the model might lead to divergence.

To prove the well posedness of the classical Naghdi shell model, the following equivalence was established in [11].

$$\|\underset{\approx}{\rho}^N(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\sim}{L}_2(\omega)}$$
$$\simeq \|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1(\omega)} + \|\underset{\sim}{u}\|_{\underset{\sim}{H}^1(\omega)} + \|w\|_{H^1(\omega)} \quad \forall\ (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H,$$

from which, by the observation

$$\|\underset{\approx}{\rho}(\underset{\sim}{\theta})\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + (1+|b|)\|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\sim}{L}_2(\omega)}$$
$$\gtrsim \|\underset{\approx}{\rho}^N(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\sim}{L}_2(\omega)},$$

the following equivalency easily follows:

$$\|\underset{\approx}{\rho}(\underset{\sim}{\theta})\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_{\underset{\sim}{L}_2(\omega)}$$
$$\simeq \|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1(\omega)} + \|\underset{\sim}{u}\|_{\underset{\sim}{H}^1(\omega)} + \|w\|_{H^1(\omega)} \quad \forall\ (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H. \quad (5.3.5)$$

Since the elastic tensor $a^{\alpha\beta\lambda\gamma}$ and the contravariant metric tensor of the middle surface $a^{\alpha\beta}$ are uniformly positive definite and bounded, the bilinear form in the left hand side of the variational equation (5.3.2) is continuous and uniformly elliptic over the space H. Therefore, we have

THEOREM 5.3.1. *If the resultant loading functionals (5.3.3) and (5.3.4) are linear continuous functionals on the space* $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$, *the model (5.3.2) has a unique solution* $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ *in this space.*

REMARK 5.3.2. *The condition on the loading functionals for the existence of the model solution can be met, if, say, the loading functions satisfy the conditions*

$$\tilde{p}^3_\pm \in L_2(\omega), \quad q^i_a \in L_2(\omega), \quad \tilde{p}^\alpha_\pm \in \underset{\sim}{H}(\mathrm{div}, \omega), \quad p^i_a,\ p^i_m \in H^{-1/2}(\partial_T \omega). \tag{5.3.6}$$

For simplicity, the flexural, membrane, and shear strains engendered by the model solution will be denoted by

$$\rho^\epsilon_{\alpha\beta} = \rho_{\alpha\beta}(\underset{\sim}{\theta}^\epsilon), \quad \gamma^\epsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\underset{\sim}{u}^\epsilon, w^\epsilon), \quad \tau^\epsilon_\alpha = \tau_\alpha(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon).$$

5.4 Reconstruction of the admissible stress and displacement fields

From the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$, we can reconstruct a statically admissible stress field and a kinematically admissible displacement field by explicitly giving their components, and compute the constitutive residual so that we can use the two energies principle to bound the error of the model solution in the energy norm. We will see that the constitutive residual is formally small. A rigorous justification, which crucially hinges on the asymptotic behavior of the model solution, will be given in the next section.

5.4.1 The admissible stress and displacement fields

Based on the model solution $(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \in H$, we define the following 2D tensors $\sigma_0^{\alpha\beta}$, $\sigma_1^{\alpha\beta}$, and a 2D vector $\sigma_0^{3\alpha}$ by

$$
\begin{aligned}
\sigma_0^{\alpha\beta} &= a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda} p_o^3 a^{\alpha\beta}, \\
\sigma_1^{\alpha\beta} &= a^{\alpha\beta\lambda\gamma}\rho_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda}(p_e^3 + b p_o^3) a^{\alpha\beta}, \\
\sigma_0^{3\alpha} &= \frac{5}{4}[\mu a^{\alpha\beta}\tau_{\beta}^{\epsilon} - p_o^{\alpha}].
\end{aligned} \tag{5.4.1}
$$

By using the model equation, it is readily checked that, in weak sense, these tensor- and vector-valued functions satisfy the following system of differential equations.

$$
\begin{aligned}
\frac{1}{3}\epsilon^2 \sigma_1^{\alpha\beta}|_{\beta} - \frac{2}{3}\sigma_0^{3\alpha} &= \epsilon^2 b p_e^{\alpha} + \frac{2}{3} b \epsilon^2 q_a^{\alpha}, \\
\sigma_0^{\alpha\beta}|_{\beta} - \frac{2}{3} b \sigma_0^{3\alpha} &= 2 b p_o^{\alpha} - p_e^{\alpha} - (1 + \frac{1}{3} b^2 \epsilon^2) q_a^{\alpha}, \\
b a_{\alpha\beta}\sigma_0^{\alpha\beta} + \frac{2}{3}\sigma_0^{3\alpha}|_{\alpha} &= -p_o^{\alpha}|_{\alpha} - p_e^3 - (1 + \frac{1}{3} b \epsilon^2) q_a^3
\end{aligned} \tag{5.4.2}
$$

and the boundary conditions on γ_T

$$
\sigma_0^{\alpha\beta} n_{\beta} = p_a^{\alpha} - \frac{1}{3} b \epsilon^2 p_m^{\alpha}, \quad \sigma_1^{\alpha\beta} n_{\beta} = p_m^{\alpha} - b p_a^{\alpha}, \quad \sigma_0^{3\alpha} n_{\alpha} = \epsilon^2 b p_m^3. \tag{5.4.3}
$$

Indeed, if we substitute (5.4.1) into the above equations and boundary conditions, and write the resulting equations and boundary conditions in weak form, we will get the model equation (5.3.2). This is in fact how the model was derived.

The functions $\sigma_0^{\alpha\beta}$, $\sigma_1^{\alpha\beta}$ and $\sigma_0^{3\alpha}$ furnish the principal part of the statically admissible stress field. To complete the construction of the stress field, we need to define another 2D tensor-valued function $\sigma_2^{\alpha\beta}$ and two scalar valued functions σ_0^{33} and σ_1^{33}. The tensor-valued function $\sigma_2^{\alpha\beta}$ will be determined by the equation

$$\sigma_2^{\alpha\beta}|_\beta = -4b\sigma_0^{3\alpha} - b^2\,\epsilon^2\,q_a^\alpha \tag{5.4.4}$$

and the boundary condition

$$\sigma_2^{\alpha\beta} n_\beta = -\,\epsilon^2\,bp_m^\alpha \quad \text{on } \gamma_T. \tag{5.4.5}$$

This equation and boundary condition together do not uniquely determine $\sigma_2^{\alpha\beta}$. We will choose one so that

$$\|\sigma_2^{\alpha\beta}\|_{\underset{\approx}{L}^2(\omega)} \lesssim |b|\|\sigma_0^{3\alpha}\|_{\underset{\sim}{L}^2(\omega)} + b^2\,\epsilon^2\,\|q_a^\alpha\|_{\underset{\sim}{L}^2(\omega)} + |b|\,\epsilon^2\,\|p_m^\alpha\|_{\underset{\sim}{H}^{-1/2}(\partial_T\omega)} \tag{5.4.6}$$

holds. This is possible in view of Theorem 6.3.1 below.

The other two scalar functions σ_0^{33} and σ_1^{33} are explicitly defined by

$$\begin{aligned} \sigma_0^{33} &= \frac{1}{2}\,\epsilon^2(ba_{\alpha\beta}\sigma_1^{\alpha\beta} + p_e^\alpha|_\alpha - 2bq_a^3), \\ \sigma_1^{33} &= \frac{1}{2}\,\epsilon[ba_{\alpha\beta}\sigma_0^{\alpha\beta} + \frac{2}{3}ba_{\alpha\beta}\sigma_2^{\alpha\beta} + p_o^\alpha|_\alpha + p_e^3 + (1+b^2\,\epsilon^2)q_a^3]. \end{aligned} \tag{5.4.7}$$

With all these tensor-, vector-, and scalar-valued 2D functions determined, we define the rescaled components $\tilde{\sigma}^{ij}$ of the stress field by

$$\begin{aligned}\tilde{\sigma}^{\alpha\beta} &= \sigma_0^{\alpha\beta} + t\sigma_1^{\alpha\beta} + r(t)\sigma_2^{\alpha\beta},\\ \tilde{\sigma}^{3\alpha} = \tilde{\sigma}^{\alpha 3} &= p_o^{\alpha} + tp_e^{\alpha} + q(t)\sigma_0^{3\alpha},\\ \tilde{\sigma}^{33} &= p_o^3 + tp_e^3 + q(t)\sigma_0^{33} + s(t)\sigma_1^{33},\end{aligned} \tag{5.4.8}$$

where $r(t)$, $q(t)$, and $s(t)$ were defined in (2.4.6). From the definition, it is obvious that the surface force conditions (5.2.6) on the upper and lower surfaces are precisely satisfied. By using the boundary conditions (5.4.3), (5.4.5), and the compatibility condition

$$p_o^{\alpha} n_{\alpha} = p_a^3 - \epsilon^2 bp_m^3, \quad p_e^{\alpha} n_{\alpha} = p_m^3 - bp_a^3 \quad \text{on} \quad \gamma_T, \tag{5.4.9}$$

we can verify that the lateral surface force condition (5.2.7) is also exactly satisfied.

By using the equation (5.4.2), (5.4.4), and the definition (5.4.7), after a straightforward calculation, we can verify that the equilibrium equation (5.2.5) is precisely satisfied by the constructed rescaled stress components. Therefore, the functions $\tilde{\sigma}^{ij}$ defined by (5.4.8) are the rescaled components of a statically admissible stress field $\boldsymbol{\sigma}$, whose contravariant components, by the relation (5.2.2), are given by

$$\sigma^{\alpha\beta} = \frac{1}{(1-bt)^3}[\sigma_0^{\alpha\beta} + t\sigma_1^{\alpha\beta} + r(t)\sigma_2^{\alpha\beta}],$$
$$\sigma^{3\alpha} = \sigma^{\alpha 3} = \frac{1}{(1-bt)^2}[p_o^{\alpha} + tp_e^{\alpha} + q(t)\sigma_0^{3\alpha}], \tag{5.4.10}$$
$$\sigma^{33} = \frac{1}{(1-bt)^2}[p_o^3 + tp_e^3 + q(t)\sigma_0^{33} + s(t)\sigma_1^{33}].$$

REMARK 5.4.1. *On the shell edges $\Gamma_+ \cap \Gamma_T$ and $\Gamma_- \cap \Gamma_T$, where the upper and lower surfaces meet the lateral surface, the surface forces exerted on the upper and lower surfaces must be compatible with the force applied on the lateral surface in the sense that*

$$\boldsymbol{p}_+ \cdot \boldsymbol{n}^*(\epsilon) = \boldsymbol{p}_T \cdot \boldsymbol{g}_3 \text{ on } \Gamma_+ \cap \Gamma_T, \quad \boldsymbol{p}_- \cdot \boldsymbol{n}^*(-\epsilon) = -\boldsymbol{p}_T \cdot \boldsymbol{g}_3 \text{ on } \Gamma_- \cap \Gamma_T.$$

This compatibility condition is precisely equivalent to (5.4.9).

The kinematically admissible displacement field $\boldsymbol{v}$ is defined by giving its rescaled components as

$$\tilde{v}_\alpha = u_\alpha^\epsilon + t\theta_\alpha^\epsilon, \quad \tilde{v}_3 = w^\epsilon + tw_1, \tag{5.4.11}$$

in which $w_1 \in H_D(\omega)$ is a correction function to the transverse deflection whose definition will be given in the next section. The clamping boundary condition on Γ_D is obviously satisfied.

5.4.2 The constitutive residual

For the admissible stress field $\boldsymbol{\sigma}$ and displacement field $\boldsymbol{v}$ constructed in the previous subsection, we denote the residual of the constitutive equation by $\varrho_{ij} = A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})$. By Lemma 4.3.2, the covariant components of strain tensor $\chi_{ij}(\boldsymbol{v})$ engendered by the displacement $\boldsymbol{v}$ defined in (5.4.11) are

$$\chi_{\alpha\beta}(\boldsymbol{v}) = (1-bt)(\gamma^{\epsilon}_{\alpha\beta} + t\rho^{\epsilon}_{\alpha\beta}) - bt(1-bt)w_1 a_{\alpha\beta},$$

$$\chi_{3\alpha}(\boldsymbol{v}) = \chi_{\alpha 3}(\boldsymbol{v}) = \frac{1}{2}\tau^{\epsilon}_{\alpha} + \frac{1}{2}t\partial_\alpha w_1, \quad \chi_{33}(\boldsymbol{v}) = w_1. \tag{5.4.12}$$

For the admissible stress field $\boldsymbol{\sigma}$ defined by (5.4.10), we can compute $A_{ijkl}\sigma^{kl}$ by using the definition of the 3D compliance tensor, the relations (5.2.1), and the definition (5.4.1). The results are

$$\begin{aligned} A_{\alpha\beta kl}\sigma^{kl} &= (1-bt)(\gamma^{\epsilon}_{\alpha\beta} + t\rho^{\epsilon}_{\alpha\beta}) - bt^2\frac{\lambda}{2\mu(2\mu+3\lambda)}(p^3_e + bp^3_o)a_{\alpha\beta} \\ &\quad - \frac{\lambda}{2\mu(2\mu+3\lambda)}[q(t)\sigma^{33}_0 + s(t)\sigma^{33}_1]a_{\alpha\beta} \\ &\quad + (1-bt)r(t)\frac{1}{2\mu}[a_{\alpha\lambda}a_{\beta\gamma} - \frac{\lambda}{2\mu+3\lambda}a_{\alpha\beta}a_{\lambda\gamma}]\sigma^{\lambda\gamma}_2, \end{aligned} \tag{5.4.13}$$

$$A_{3\alpha kl}\sigma^{kl} = \frac{1}{2\mu}a_{\alpha\beta}[p^\beta_o + tp^\beta_e + q(t)\sigma^{3\beta}_0],$$

$$A_{33kl}\sigma^{kl} = \frac{1}{2\mu(1-bt)^2}\{\frac{2(\mu+\lambda)}{2\mu+3\lambda}[p^3_o + tp^3_e + q(t)\sigma^{33}_0 + s(t)\sigma^{33}_1]$$

$$-\frac{\lambda}{2\mu+3\lambda}(1-bt)[\sigma_0^{\alpha\beta}+t\sigma_1^{\alpha\beta}+r(t)\sigma_2^{\alpha\beta}]a_{\alpha\beta}\}.$$

Subtracting (5.4.12) from (5.4.13), we get the explicit expression of the residual ϱ_{ij}:

$$\begin{aligned}
\varrho_{\alpha\beta} &= -bt^2\frac{\lambda}{2\mu(2\mu+3\lambda)}(p_e^3+bp_o^3)a_{\alpha\beta} \\
&\quad -\frac{\lambda}{2\mu(2\mu+3\lambda)}[q(t)\sigma_0^{33}+s(t)\sigma_1^{33}]a_{\alpha\beta} \\
&\quad +(1-bt)r(t)\frac{1}{2\mu}[a_{\alpha\lambda}a_{\beta\gamma}-\frac{\lambda}{2\mu+3\lambda}a_{\alpha\beta}a_{\lambda\gamma}]\sigma_2^{\lambda\gamma} \\
&\quad +b(1-bt)tw_1a_{\alpha\beta},
\end{aligned}$$

$$\varrho_{3\alpha} = \frac{1}{2\mu}[q(t)-\frac{4}{5}]a_{\alpha\beta}\sigma_0^{3\beta}-t\frac{1}{2\mu}a_{\alpha\beta}p_e^\beta-\frac{1}{2}t\partial_\alpha w_1, \tag{5.4.14}$$

$$\begin{aligned}
\varrho_{33} &= \frac{1}{(1-bt)^2}(\frac{1}{2\mu+\lambda}p_o^3-\frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}\gamma_{\alpha\beta}^\epsilon)-w_1 \\
&\quad +\frac{bt}{(1-bt)^2}\frac{\lambda}{2\mu(2\mu+3\lambda)}a_{\alpha\beta}\sigma_0^{\alpha\beta} \\
&\quad +\frac{1}{2\mu(1-bt)^2}[\frac{2(\mu+\lambda)}{2\mu+3\lambda}(tp_e^3+q(t)\sigma_0^{33}+s(t)\sigma_1^{33}) \\
&\quad -\frac{\lambda}{2\mu+3\lambda}(1-bt)(t\sigma_1^{\alpha\beta}+r(t)\sigma_2^{\alpha\beta})a_{\alpha\beta}].
\end{aligned}$$

In the next section, we will prove that under some assumptions,

$$\sigma_0^{3\alpha}=\frac{5}{4}[\mu a^{\alpha\beta}\tau_\beta^\epsilon-p_o^\alpha]\to 0$$

as $\epsilon \to 0$. By the estimate (5.4.6), we know that $\sigma_2^{\alpha\beta}$ will converge to zero. From the definition (5.4.7) we know that σ_0^{33} and σ_1^{33} are formally small. To make ϱ_{33} small, we will choose $w_1 \in H_D^1(\omega)$ to minimize

$$[\frac{1}{2\mu+\lambda} p_o^3 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta} \gamma_{\alpha\beta}(\underset{\sim}{u}^{\epsilon}, w^{\epsilon})] - w_1. \tag{5.4.15}$$

At the same time, due to the involvement of $t\partial_\alpha w_1$ in the expression of $\varrho_{3\alpha}$, the quantity $\epsilon \|w_1\|_{H^1(\omega)}$ needs to be small. With all these considerations, we can expect the constitutive residual to be small.

5.5 Justification

The formal observations we made in the last subsection do not furnish a rigorous justification, since the applied forces and the model solution may depend on the shell thickness in an unexpected way. To prove the convergence, we need to make some assumptions on the applied loads, and have a good grasp of the behavior of the model solution when the shell thickness tends to zero. When $\epsilon \to 0$, everything may tend to zero, so to prove the convergence, we need to consider the relative error. In addition to the upper bound that can be obtained by bounding the constitutive residual, we need to have a lower bound on the model solution.

5.5.1 Assumption on the applied forces

Henceforth, we assume that all the applied force functions explicitly involved in the resultant loading functional in the model are independent of ϵ. I.e., we assume that the functions

$$p_o^i,\ p_e^i,\ q_a^i,\ p_a^i,\ p_m^i \text{ are independent of } \epsilon. \tag{5.5.1}$$

This assumption is different from the usual assumption adopted in asymptotic theories, according to which, the functions $\epsilon^{-1} p_o^i$, rather than p_o^i themselves, should have been assumed to be independent of ϵ. Our assumption on p_e^i and q_a^i is the same as the usual assumption [18].

Our assumption will reveal the potential advantages of using the Naghdi-type model over the Koiter-type model. The convergence theorem can also be proved under the usual assumption on the applied forces, but in that case, the difference between the two types of models is negligible.

5.5.2 Asymptotic behavior of the model solution

Under the loading assumption (5.5.1), the shell model (5.3.2) fits into the abstract ϵ-dependent variational problem (3.2.2) of Chapter 3. To apply the abstract theory, we define the following spaces and operators. As above $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$ with the usual product norm. We let $U = \underset{\approx}{L}{}_2^{\text{sym}}(\omega)$ with the equivalent inner product

$$(\underset{\approx}{\rho}^1, \underset{\approx}{\rho}^2)_U = \frac{1}{3}\int_\omega a^{\alpha\beta\lambda\gamma} \rho^1_{\lambda\gamma} \rho^2_{\alpha\beta} \sqrt{a}\, d\underset{\sim}{x} \ \ \forall\ \underset{\approx}{\rho}^1, \underset{\approx}{\rho}^2 \in U,$$

and define $A : H \to U$, the flexural strain operator, by

$$A(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \underset{\approx}{\rho}(\underset{\sim}{\theta}) \ \ \forall\ (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H.$$

We also define $B : H \to \underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, combining the membrane and shear strain operators, by

$$B(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = [\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)] \ \ \forall\ (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H.$$

The equivalence (5.3.5) guaranteed the condition (3.2.1) required by the abstract theory.

A totally or partially clamped spherical shell is stiff in the sense that it does not allow for non-stretching deformations. If $\underset{\approx}{\gamma}(\underset{\sim}{u}, w) = 0$, we must have $\underset{\sim}{u} = 0$ and $w = 0$ [18]. Therefore, $\ker B = 0$. According to the classification of the abstract ϵ-dependent variational problem in Section 3.5, a spherical shell can never be a flexural shell.

For spherical shells, the most significant difference from the plane strain cylindrical shells is that the operator B does not have closed range. We need to consider the space $W = B(H) \subset \underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, in which the norm is defined by

$$\|[\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)]\|_W = \|(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_H.$$

Equipped with this norm, W is a Hilbert space isomorphic to H. The operator B is, of course, surjective from H to W.

The space V is defined as the closure of W in $\underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, with the inner product

$$((\underset{\approx}{\gamma}^1, \underset{\sim}{\tau}^1), (\underset{\approx}{\gamma}^2, \underset{\sim}{\tau}^2))_V = \int_\omega a^{\alpha\beta\lambda\gamma} \gamma^1_{\lambda\gamma} \gamma^2_{\alpha\beta} \sqrt{a} d\underset{\sim}{x} + \frac{5}{6}\mu \int_\omega a^{\alpha\beta} \tau^1_\beta \tau^2_\alpha \sqrt{a} d\underset{\sim}{x},$$

which is equivalent to the inner product of $\underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$.

The range of the operator B then is dense in V, as was required by the abstract theory. The space V actually is equal to the product of V_0, the closure of the range of membrane strain operator $\underset{\approx}{\gamma}$ in $\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)$, and the closure of the range of shear strain operator $\underset{\sim}{\tau}$ in $\underset{\sim}{L}_2(\omega)$. The latter, since the range of $\underset{\sim}{\tau}$ is dense in $\underset{\sim}{L}_2(\omega)$, is just equal to $\underset{\sim}{L}_2(\omega)$. Therefore, we have the factorization

$$V = V_0 \times \underset{\sim}{L}_2(\omega). \tag{5.5.2}$$

According to the discussions in Section 3.4, the leading resultant loading functional $\boldsymbol{f}_0$ determines the asymptotic behavior of the model solution.

Since $\ker B = 0$ and B is an onto mapping from H to W, by the closed range theorem, there exists a $\zeta_*^0 \in W^*$, such that the leading term in the loading functional can be equivalently written as

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \rangle = \langle \zeta_*^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z) \rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H.$$

We recall that without further assumption, the solution of this essentially singular perturbation problem is untractable. To sort out the tractable situations, we imposed the condition (3.3.24) on ζ_*^0 in Chapter 3. Namely,

$$\zeta_*^0 \in V^*. \tag{5.5.3}$$

This condition is equivalent to the requirement that the loading functional can be written as

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \langle \zeta^0_*, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle_{V^*\times V} = (\zeta^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z))_V,$$

here $\zeta^0 \in V$ is the Riesz representation of $\zeta^0_* \in V^*$. Therefore the condition (5.5.3) is equivalently requiring the existence of $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V = V_0 \times \underset{\sim}{L}_2(\omega)$ such that

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \int_\omega a^{\alpha\beta\lambda\gamma}\gamma^0_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu\int_\omega a^{\alpha\beta}\tau^0_\beta\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}. \quad (5.5.4)$$

Recalling the expression of the leading loading functional

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{5}{6}\int_\omega p^\alpha_o\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} - \frac{\lambda}{2\mu+\lambda}\int_\omega p^3_o a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$+ \int_\omega [(q^\alpha_a - 2bp^\alpha_o + p^\alpha_e)y_\alpha + (q^3_a + p^\alpha_o|_\alpha + p^3_e)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p^\alpha_a y_\alpha, \quad (5.5.5)$$

we can see that the condition (5.5.4) is equivalent to the existence of $\underset{\approx}{\kappa} \in V_0$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\kappa_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$= \int_\omega [(q^\alpha_a - 2bp^\alpha_o + p^\alpha_e)y_\alpha + (q^3_a + p^\alpha_o|_\alpha + p^3_e)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p^\alpha_a y_\alpha$$
$$\forall\ (\underset{\sim}{y}, z) \in \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega). \quad (5.5.6)$$

Note that the second term in (5.5.5) can be equally written as

$$-\frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x} = -\frac{\lambda}{2\mu(2\mu+3\lambda)}\int_\omega a^{\alpha\beta\lambda\gamma}a_{\lambda\gamma}p_o^3\gamma_{\alpha\beta}(\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x}, \tag{5.5.7}$$

so if $p_o^3 \in L_2(\omega)$, we can determine $\underset{\approx}{\gamma}^0 \in V_0$ as

$$\gamma^0_{\alpha\beta} = \kappa_{\alpha\beta} - \frac{\lambda}{2\mu(2\mu+3\lambda)}P_{V_0}(a_{\alpha\beta}p_o^3), \tag{5.5.8}$$

where P_{V_0} is the orthogonal projection from $\underset{\approx}{L}_2^{\text{sym}}(\omega)$ to its closed subspace V_0, with respect to the inner product of U.

By defining

$$\tau^0_\alpha = \frac{1}{\mu}a_{\alpha\beta}p_o^\beta, \tag{5.5.9}$$

we obtain $\zeta^0 = (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V$ such that the loading functional can be reformulated as

$$\begin{aligned}\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle_{H^*\times H} &= ((\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), B(\underset{\sim}{\phi}, \underset{\sim}{y}, z))_V \\ &= \int_\omega a^{\alpha\beta\lambda\gamma}\gamma^0_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu\int_\omega a^{\alpha\beta}\tau^0_\beta\tau_\alpha(\underset{\sim}{\phi},\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x}.\end{aligned} \tag{5.5.10}$$

Therefore, to use the abstract theory, the crux is the existence of $\underset{\approx}{\kappa} \in V_0$, such that the problem (5.5.6) is solvable. We will see that for totally clamped spherical shells, under the data assumption (5.3.6), the existence of $\underset{\approx}{\kappa} \in V_0$ is guaranteed automatically. But for partially clamped spherical shells, this requirement imposes a stringent restriction on the applied forces. Even if the shell data are infinitely smooth, this requirement might not be met.

Under the condition (5.5.6), the asymptotic behavior of the model solution follows from Theorem 3.3.5 and (3.4.7). We have the convergence

$$\lim_{\epsilon\to 0}[\epsilon\,\|\underset{\approx}{\rho}^{\epsilon}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}+\|\underset{\approx}{\gamma}^{\epsilon}-\underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}+\|\underset{\sim}{\tau}^{\epsilon}-\underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}{}_2(\omega)}]=0. \tag{5.5.11}$$

From this, we get the estimates

$$\|\underset{\approx}{\rho}^{\epsilon}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}\lesssim o(\epsilon^{-1}),\quad \|\underset{\approx}{\gamma}^{\epsilon}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}\lesssim 1,\quad \|\underset{\sim}{\tau}^{\epsilon}\|_{\underset{\sim}{L}{}_2(\omega)}\lesssim 1.$$

From the equivalency (5.3.5), we get the *a priori* estimates on the model solution

$$\begin{gathered}\|\underset{\sim}{\theta}^{\epsilon}\|_{\underset{\sim}{H}{}^1(\omega)}+\|\underset{\sim}{u}^{\epsilon}\|_{\underset{\sim}{H}{}^1(\omega)}\lesssim o(\epsilon^{-1}),\\ \|w^{\epsilon}\|_{H^1(\omega)}\lesssim\|\underset{\sim}{\theta}^{\epsilon}\|_{\underset{\sim}{L}{}_2(\omega)}+\|\underset{\sim}{u}^{\epsilon}\|_{\underset{\sim}{L}{}_2(\omega)}.\end{gathered} \tag{5.5.12}$$

If we assume more regularity on $(\underset{\approx}{\gamma}^{0},\underset{\sim}{\tau}^{0})$, say,

$$(\underset{\approx}{\gamma}^{0},\underset{\sim}{\tau}^{0})\in[W,V]_{1-\theta,q}, \tag{5.5.13}$$

for some $\theta\in(0,1)$ and $q\in[1,\infty]$, or $\theta\in[0,1]$ and $q\in(1,\infty)$, according to Theorem 3.3.4 and (3.4.6), we have the stronger estimate on the asymptotic behavior of the model solution:

$$\epsilon\,\|\underset{\approx}{\rho}^{\epsilon}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}+\|\underset{\approx}{\gamma}^{\epsilon}-\underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)}+\|\underset{\sim}{\tau}^{\epsilon}-\underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}{}_2(\omega)}\lesssim K(\epsilon,(\underset{\approx}{\gamma}^{0},\underset{\sim}{\tau}^{0}),[V,W])\lesssim\epsilon^{\theta}. \tag{5.5.14}$$

And the estimates

$$\|\underset{\approx}{\rho}^{\epsilon}\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} \lesssim \epsilon^{\theta-1}, \quad \|\underset{\approx}{\gamma}^{\epsilon}\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} \lesssim 1, \quad \|\underset{\sim}{\tau}^{\epsilon}\|_{\underset{\sim}{L}_2(\omega)} \lesssim 1.$$

By the equivalency (5.3.5), we get the *a priori* estimates

$$\begin{gathered} \|\underset{\sim}{\theta}^{\epsilon}\|_{\underset{\sim}{H}^1(\omega)} + \|\underset{\sim}{u}^{\epsilon}\|_{\underset{\sim}{H}^1(\omega)} \lesssim \epsilon^{\theta-1}, \\ \|w^{\epsilon}\|_{H^1(\omega)} \lesssim \|\underset{\sim}{\theta}^{\epsilon}\|_{\underset{\sim}{L}_2(\omega)} + \|\underset{\sim}{u}^{\epsilon}\|_{\underset{\sim}{L}_2(\omega)}. \end{gathered} \tag{5.5.15}$$

The correction function w_1, based on its involvements in the constitutive residual, will be defined as the solution of the variational equation

$$\begin{gathered} \epsilon^2(\nabla w_1, \nabla v)_{\underset{\sim}{L}_2(\omega)} + (w_1, v)_{L_2(\omega)} = (\frac{1}{2\mu+\lambda} p_o^3 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta}, v)_{L_2(\omega)}, \\ w_1 \in H^1_D(\omega), \quad \forall\, v \in H^1_D(\omega). \end{gathered} \tag{5.5.16}$$

Note that this definition of the correction function is not a simple analogue of the definition of w_1 in the plane strain cylindrical shell problems. Here we use $\underset{\approx}{\gamma}^0$, rather than $\underset{\approx}{\gamma}^{\epsilon}$ to define the correction. Due to the possible boundary layer of $\underset{\approx}{\gamma}^{\epsilon}$, if we put $\underset{\approx}{\gamma}^{\epsilon}$ in the place of $\underset{\approx}{\gamma}^0$ in (5.5.16), the convergence rate of the model solution will be substantially reduced. Our correction on the transverse deflection is not an *a posteriori* correction.

From the definition (5.5.8) of $\underset{\approx}{\gamma}^0$, we know $\underset{\approx}{\gamma}^0 \in \underset{\approx}{L}_2^{\text{sym}}(\omega)$, so we have $a^{\alpha\beta}\gamma^0_{\alpha\beta} \in L_2(\omega)$. By (3.3.38) in Theorem 3.3.6, we have

$$\epsilon \|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o \|_{L_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{5.5.17}$$

If we assume

$$\lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o \in [H^1_D(\omega), L_2(\omega)]_{1-\theta,p} \tag{5.5.18}$$

for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or $\theta \in [0,1]$ and $p \in (1,\infty)$, by (3.3.36) in Theorem 3.3.6, we have

$$\epsilon \|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o \|_{L_2(\omega)}$$
$$\lesssim K(\epsilon, \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_D(\omega)]) \lesssim \epsilon^\theta . \tag{5.5.19}$$

5.5.3 Convergence theorem

With the estimates on the asymptotic behavior of the model solution established in the previous subsection, and the expression of the constitutive residual (5.4.14), we are ready to prove the convergence theorem. We denote the energy norms of a stress field $\boldsymbol{\sigma}$ and a strain field $\boldsymbol{\chi}$ by on the shell Ω^ϵ by

$$\|\boldsymbol{\sigma}\|_{E^\epsilon} = (\int_{\Omega^\epsilon} A_{ijkl}\sigma^{kl}\sigma^{ij})^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{\chi}\|_{E^\epsilon} = (\int_{\Omega^\epsilon} C^{ijkl}\chi_{kl}\chi_{ij})^{\frac{1}{2}},$$

respectively. Since the elastic tensor C^{ijkl} and the compliance tensor A_{ijkl} are uniformly positive definite and bounded, the energy norms are equivalent to the sums of the $L_2(\omega^\epsilon)$ norms of the tensor components.

THEOREM 5.5.1. *Let $\boldsymbol{v}^*$ and $\boldsymbol{\sigma}^*$ be the displacement and stress fields on the spherical shell arising in response to the applied forces and boundary conditions determined from the 3D elasticity equations. And let $\boldsymbol{v}$ be the kinematically admissible displacement field defined by (5.4.11) based on the model the solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ and the correction functions w_1 defined in (5.5.16), $\boldsymbol{\sigma}$ the statically admissible stress field defined by (5.4.10).*

If there exists a $\underset{\approx}{\kappa} \in V_0$ such that the functional reformulation (5.5.6) holds, then we have the convergence

$$\lim_{\epsilon \to 0} \frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0. \tag{5.5.20}$$

If we further have the regularity (5.5.13) and (5.5.18), we have the convergence rate

$$\frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^\theta\,. \tag{5.5.21}$$

Proof. We give the proof of (5.5.21). The proof of (5.5.20) is similar. For brevity, the norm $\|\cdot\|_{L_2(\omega^\epsilon)}$ will be denoted by $\|\cdot\|$. Any function defined on ω will be viewed as a function, constant in t, defined on ω^ϵ.

First, we establish the lower bound for $\|\boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon}$. By the convergence (5.5.14), we have

$$\epsilon \, \|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^\theta, \quad \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^\theta, \quad \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^\theta. \tag{5.5.22}$$

Since $\underset{\approx}{\gamma}^0$ and $\underset{\sim}{\tau}^0$ can not be zero at the same time (otherwise $\boldsymbol{f}_0 = 0$), we have

$$\|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \simeq \|\underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \simeq 1. \tag{5.5.23}$$

By the equivalence (5.3.5), we have $\epsilon \, \|(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)\|_{\underset{\sim}{H}^1(\omega)\times \underset{\sim}{H}^1(\omega)\times H^1(\omega)} \lesssim \epsilon^\theta$. The convergence (5.5.17) shows that

$$\epsilon \, \|w_1\|_{H^1(\omega)} \lesssim \epsilon^\theta \quad \text{and} \quad \|w_1\|_{L_2(\omega)} \simeq \|\lambda a^{\alpha\beta} \gamma^0_{\alpha\beta} - p^3_o\|_{L_2(\omega)}.$$

With all these estimates, it is easy to see that in the expression (5.4.12) of $\chi_{ij}(\boldsymbol{v})$, the terms $\gamma^\epsilon_{\alpha\beta}$ and τ^ϵ_α dominate in $\chi_{\alpha\beta}$ and $\chi_{3\alpha}$ respectively, therefore,

$$\sum_{\alpha,\beta=1}^{2} \|\chi_{\alpha\beta}(\boldsymbol{v})\|^2 + \sum_{\alpha=1}^{2} \|\chi_{3\alpha}(\boldsymbol{v})\|^2 \gtrsim \epsilon (\|\underset{\approx}{\gamma}^\epsilon\|^2_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon\|^2_{\underset{\sim}{L}_2(\omega)}) \gtrsim \epsilon,$$

so,

$$\|\boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon} \gtrsim \epsilon \,. \tag{5.5.24}$$

From the two energies principle, we have

$$\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon}^2 + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}^2 = \int_{\Omega^\epsilon} A^{ijkl} \varrho_{kl} \varrho_{ij} \lesssim \sum_{i,j=1}^{3} \|\varrho_{ij}\|^2. \tag{5.5.25}$$

From the definition (5.4.1) and the definition of τ_β^0, we have

$$\sigma_0^{\alpha\beta} = a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}^\epsilon + \frac{\lambda}{2\mu+\lambda} p_o^3 a^{\alpha\beta},$$

$$\sigma_1^{\alpha\beta} = a^{\alpha\beta\lambda\gamma}\rho_{\lambda\gamma}^\epsilon + \frac{\lambda}{2\mu+\lambda}(p_e^3 + bp_o^3) a^{\alpha\beta},$$

$$\sigma_0^{3\alpha} = \frac{5}{4}[\mu a^{\alpha\beta}(\tau_\beta^\epsilon - \tau_\beta^0)],$$

and so, by (5.5.22), we have the estimates

$$\epsilon^2 \|\sigma_1^{\alpha\beta}\|^2 \lesssim \epsilon^{1+2\theta}, \quad \|\sigma_0^{\alpha\beta}\|^2 \lesssim \epsilon, \quad \|\sigma_0^{3\alpha}\|^2 \lesssim \epsilon^{1+2\theta}. \tag{5.5.26}$$

By the estimate (5.4.6), we get $\|\sigma_2^{\alpha\beta}\|^2 \lesssim \epsilon^{1+2\theta}$. From the last two equations of (5.4.7), we see $\|\sigma_0^{33}\|^2 \lesssim \epsilon^{3+2\theta}, \quad \|\sigma_1^{33}\|^2 \lesssim \epsilon^3$. Apply all the above estimations to the expression of $\varrho_{\alpha\beta}$, it is readily seen that the square integral over ω^ϵ of every term is bounded by $O(\epsilon^3)$, except the one in the third line, whose square integral on ω^ϵ is bounded by $O(\epsilon^{1+2\theta})$. Therefore we have $\|\varrho_{\alpha\beta}\|^2 \lesssim \epsilon^{1+2\theta}$.

From the convergence (5.5.19), we know $\epsilon \|w_1\|_{H^1(\omega)} \lesssim \epsilon^\theta$, so $\|t\partial_\alpha w_1\|^2 \lesssim \epsilon^{1+2\theta}$, together with (5.5.26), we have $\|\varrho_{3\alpha}\|^2 \lesssim \epsilon^{1+2\theta}$.

Our last concern is about ϱ_{33}. In its expression, we equally write the first line as

$$\frac{1}{(1-bt)^2}[\frac{1}{2\mu+\lambda}p_o^3 - \frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}\gamma^{\epsilon}_{\alpha\beta}] - w_1$$
$$= \frac{1}{(1-bt)^2}(\frac{1}{2\mu+\lambda}p_o^3 - \frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}\gamma^{0}_{\alpha\beta} - w_1)$$
$$- \frac{1}{(1-bt)^2}\frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}(\gamma^{\epsilon}_{\alpha\beta} - \gamma^{0}_{\alpha\beta}) + \frac{2bt-b^2t^2}{(1-bt)^2}w_1.$$

By the convergence (5.5.22) and (5.5.19), wee see that the square integral of this expression is bounded by $O(\epsilon^{1+2\theta})$. The second and third lines are obviously bounded by $O(\epsilon^3)$. The last line is bounded by $O(\epsilon^{1+2\theta})$, so, we have $\|\varrho_{33}\|^2 \lesssim \epsilon^{1+2\theta}$. We obtained the upper bound $\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|^2_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon} \lesssim \epsilon^{1+2\theta}$.

The estimate (5.5.21) follows from the lower bound (5.5.24) and this upper bound. The proof of (5.5.20) is a verbatim repetition, except replacing $\epsilon^{1+2\theta}$ by $o(\epsilon)$, and ϵ^θ by $o(1)$. □

REMARK 5.5.1. *If the odd part of the applied tangential surface forces p_o^α is not zero, the deformation violates the Kirchhoff–Love hypothesis. Actually, the estimate (5.5.22) shows that the transverse shear strain converges to the finite limit $\frac{1}{\mu}a_{\alpha\beta}p_o^\beta$.*

5.5.4 About the condition of the convergence theorem

The convergence theorem hinges on the existence of $\underset{\approx}{\kappa} \in V_0$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\kappa_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$= \int_\omega [(q_a^\alpha - 2bp_o^\alpha + p_e^\alpha)y_\alpha + (q_a^3 + p_o^\alpha|_\alpha + p_e^3)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p_a^\alpha y_\alpha$$
$$\forall\ (\underset{\sim}{y}, z) \in \underset{\sim}{H}_D^1(\omega) \times H_D^1(\omega). \quad (5.5.27)$$

The membrane strain operator $\underset{\approx}{\gamma}(\underset{\sim}{y}, z)$ defines a linear continuous operator $\underset{\approx}{\gamma} : \underset{\sim}{H}_D^1(\omega) \times H_D^1(\omega) \longrightarrow V_0$, whose range is dense in V_0. Since $\ker \underset{\approx}{\gamma} = 0$, the function $\|\underset{\approx}{\gamma}(\underset{\sim}{y}, z)\|_{\underset{\approx}{L}_2^{\text{sym}}}$ defines a norm on the space $\underset{\sim}{H}_D^1(\omega) \times H_D^1(\omega)$, which is weaker than the original norm. In the notation of [18], we denote the completion of $\underset{\sim}{H}_D^1(\omega) \times H_D^1(\omega)$ with respect to this new norm by $V_M^\sharp(\omega)$. Obviously, $\underset{\approx}{\gamma}$ can be extended uniquely to $V_M^\sharp(\omega)$, and the extended linear continuous operator, still denoted by $\underset{\approx}{\gamma}$, defines an isomorphism between $V_M^\sharp(\omega)$ and V_0. By the closed range theorem, for any $f \in [V_M^\sharp(\omega)]^*$, there exists a unique $\underset{\approx}{\kappa} \in V_0$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\kappa_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} = \langle f, (\underset{\sim}{y}, z)\rangle_{[V_M^\sharp(\omega)]^* \times [V_M^\sharp(\omega)]} \quad \forall\ (\underset{\sim}{y}, z) \in V_M^\sharp(\omega). \quad (5.5.28)$$

Therefore, the question of existence of $\underset{\approx}{\kappa} \in V_0$ in (5.5.27) is equivalent to the question that whether or not the linear functional defined on the space $\underset{\sim}{H}_D^1(\omega) \times H_D^1(\omega)$ by the right hand side of (5.5.27) can be extended to a linear continuous functional on $V_M^\sharp(\omega)$.

Under some smoothness assumption on the boundary γ of the middle surface, the following Korn-type inequality was established in [23] and [19]: There exists a constant C such that for any $\underset{\sim}{u} \in \underset{\sim}{H}^1_0(\omega),\ w \in L_2(\omega)$

$$\|\underset{\sim}{u}\|^2_{\underset{\sim}{H}^1(\omega)} + \|w\|^2_{L_2(\omega)} \leq C\|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|^2_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)}. \tag{5.5.29}$$

Therefore, if the shell is totally clamped, by this inequality, it is easily seen that

$$V^\sharp_M(\omega) = \underset{\sim}{H}^1_0(\omega) \times L_2(\omega).$$

In this case, the mild condition (5.3.6) is enough to guarantee the existence of $\underset{\approx}{\kappa} \in V_0$, and therefore the convergence (5.5.20).

If the shell is partially clamped, the space $V^\sharp_M(\omega)$ can be huge and its norm can be very weak, so that the existence of $\underset{\approx}{\kappa}$ can not be guaranteed even if the loading functions are in $\mathcal{D}(\omega)$, the space of test functions of distribution, see [38].

As to the convergence rate, the regularity requirements (5.5.13) and (5.5.18) are quite abstract. Except for the cases in which we purposely load the shell in such a way that the conditions are satisfied, we have no idea about how to explain them for partially clamped shells.

For totally clamped spherical shells, under the smoothness assumption on the shell data: $\gamma \in C^4$, $p^\alpha_o \in H^3(\omega)$, $p^\alpha_e \in H^1(\omega)$, $p^3_o \in H^2(\omega)$, $p^3_e \in H^2(\omega)$, $q^\alpha_a \in H^1(\omega)$, $q^3_a \in H^2(\omega)$, we can prove

$$K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) \lesssim \epsilon^{1/6}$$

and

$$K(\epsilon, \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_0(\omega)]) \lesssim \epsilon^{1/2}.$$

Therefore, the regularity conditions (5.5.13) and (5.5.18) hold for $\theta = 1/6$, and so the convergence rate in (5.5.21) can be determined as $\epsilon^{1/6}$. If the odd part of the tangential surface forces vanishes, or very small, the value of θ is 1/5. See Section 6.6.4.

5.5.5 A shell example for which the model might fail

The condition $\zeta^0_* \in V^*$, or equivalently, the reformulation (5.5.10) of the leading term of the resultant loading functional, is necessary for our justification of the spherical shell model (5.3.2). As we have seen, this condition is almost trivially satisfied for a totally clamped spherical shell, but it imposes a stringent restriction on the shell data if the shell is partially clamped. We give an example, for which the the condition can not be satisfied, and so the model can not be justified.

Consider a partially clamped spherical shell not subject to any body force ($\boldsymbol{q} = 0$), or upper and lower surface forces ($\boldsymbol{p}_\pm = 0$), loaded by lateral surface force $\boldsymbol{p}_T = \epsilon^{-2}\, tM^\alpha \boldsymbol{a}_\alpha$ ($p^i_a = 0$, $p^3_m = 0$, and $p^\alpha_m = \epsilon^{-2}\, M^\alpha$). The vector valued function M^α is defined on $\partial_T\omega$, and independent of ϵ. To get the physical meaning, we can imagine applying a pure bending moment of fixed magnitude on the traction lateral surface of a sequence of spherical shells with thickness tending to zero. With this load applied on the 3D shell, the resultant loading functional in the model will be

$$\langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{1}{3}\int_{\gamma_T} M^\alpha(\phi_\alpha - by_\alpha).$$

The condition $\zeta_*^0 \in V^*$ is equivalent to, see (5.5.10), the existence of $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V_0 \times \underset{\sim}{L}_2(\omega)$, such that

$$\frac{1}{3}\int_{\gamma_T} M^\alpha(\phi_\alpha - by_\alpha) = \int_\omega a^{\alpha\beta\lambda\gamma}\gamma^0_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu\int_\omega a^{\alpha\beta}\tau^0_\beta\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$

$$\forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H.$$

Recalling the definition (5.3.1) of the operators $\gamma_{\alpha\beta}$ and τ_α, we can see that this is a condition impossible to satisfy, therefore the model (5.3.2) can not be justified for this specially loaded spherical shell. The limiting membrane shell model has no solution for this problem. Our model gives a solution in the space H, but convergence in the relative energy norm can not be proved.

Chapter 6

General shell theory

6.1 Introduction

In this chapter, we present and justify the 2D model for general 3D shells. The form of the model is similar to that of the cylindrical and spherical shell models of Chapters 2 and 5. The model is a close variant of the classical Naghdi shell model, which can be fit into the abstract ϵ-dependent variational problem of Chapter 3, and it can be accordingly classified as a flexural shell or a membrane–shear shell. By proving convergence of the 2D model solution to the 3D solution in the relative energy norm, the model is completely justified for flexural shells and totally clamped elliptic shells. The latter are special membrane–shear shells. Convergence in the relative energy norm is also proved for other membrane–shear shells under the assumption that the applied forces are “admissible”. Convergence rates are established, which are related to the shell data in an abstract notion.

For general shells, the main difficulty to overcome is that, unlike for the special shells, we can not construct a statically admissible stress field from the model solution, so the two energies principle can not be used to justify the model. As an alternative, we will reconstruct a stress field that is almost admissible, which has small residuals in the equilibrium equation and lateral traction boundary condition. We will establish an integration identity (6.3.17) to incorporate the equilibrium residual and lateral traction boundary condition residual. This identity is a substitute to the two energies principle for the analysis of general shells.

The model is justified for flexural shells in Section 6.5. In this case, convergence in the relative energy norm can be proved without any assumption. If solution of the limiting flexural model, which is an ϵ-independent problem, is assumed to have some regularity in the notion of interpolation spaces, convergence rate of the 2D model solution toward the 3D shell solution can be established. The theory will be applied to the plate bending problem, which is a special flexural shell problem, to reproduce the plate bending theory. We can use the known results for this special problem to argue that the convergence rate we determined for flexural shells is the best possible.

We justify the model for totally clamped elliptic shells in Section 6.6. The reason of sorting out these special membrane–shear shells is that totally clamped elliptic shells possess some special properties, especially the Korn-type inequality (6.6.2), so that we can prove the convergence theorem without making any assumption. Convergence rate will be determined if the solution of the limiting membrane shell model has some regularity. With some smoothness in the usual Sobolev sense assumed on the shell data, the convergence rate $O(\epsilon^{1/6})$ in the relative energy norm will be determined.

Section 6.7 is devoted to the justification of the model for all the other membrane–shear shells. In the general situation, there are more difficulties to overcome. The model justification can only be obtained under some restrictions on the applied forces. There are two sources for the new difficulties, one is rooted in the model, the other is due to the residuals of equilibrium equation and lateral traction boundary condition of the reconstructed almost admissible stress field. The first one is resolved by concreting the condition (3.3.24) that we introduced in Chapter 3 on the abstract level. The second will be resolved by adopting the condition of "admissible applied forces" proposed in [18]. Under these assumptions, the convergence of the model solution to the 3D solution in the relative energy norm will be proved.

To address the potential superiority of our Naghdi-type model over the Koiter-type model, we make an assumption on the loading functions, which is slightly different from what usually assumed in asymptotic theories. We will see that under this loading assumption, there is no significant difference between the two types of models for flexural shells. But for membrane–shear shells, it is very likely that the Koiter-type model does not converge while the Naghdi-type model does. In Chapter 7, we will show that under the usual assumption on the applied forces, the difference between these two types of models is not significant.

6.2 The shell model

The general 3D shell problem is what was described in Chapter 4. The shell Ω^ϵ is assumed to be clamped on a part of its lateral surface Γ_D. It is subjected to surface traction force on the remaining part Γ_T of the lateral surface, whose density is $\boldsymbol{p}_T$. The shell is subjected to surface forces on the upper and lower surfaces $\Gamma_\pm$, whose densities are $\boldsymbol{p}_\pm$, and loaded by a body force with density $\boldsymbol{q}$.

In terms of the rescaled surface force components $\tilde{p}^i_\pm$, see (4.3.4), we define the odd and weighted even parts of the surface forces by

$$p^\alpha_o = \frac{\tilde{p}^\alpha_+ - \tilde{p}^\alpha_-}{2}, \quad p^\alpha_e = \frac{\tilde{p}^\alpha_+ + \tilde{p}^\alpha_-}{2\,\epsilon}, \quad p^3_o = \frac{\tilde{p}^3_+ - \tilde{p}^3_-}{2}, \quad p^3_e = \frac{\tilde{p}^3_+ + \tilde{p}^3_-}{2\,\epsilon}. \qquad (6.2.1)$$

For the body force, we define the components of the transverse average by

$$q_a^i = \frac{1}{2\,\epsilon}\int_{-\,\epsilon}^{\epsilon} \boldsymbol{q}\cdot\boldsymbol{a}^i dt. \tag{6.2.2}$$

We assume the body force density is constant in the transverse coordinate. This assumption is equivalent $\boldsymbol{q} = q_a^i \boldsymbol{a}_i$. Under this assumption, the rescaled components of the body force density $\tilde{q}^i = \rho q_a^i$, see (4.3.6), are quadratic polynomials of t, and we have $\tilde{q}^i = q_0^i + tq_1^i + t^2 q_2^i$, with $q_0^i = q_a^i$, $q_1^i = -2Hq_a^i$, and $q_2^i = Kq_a^i$. The following calculation can be carried through if $\tilde{q}^i$ are arbitrary quadratic polynomials of t.

We assume that the rescaled lateral surface force components, see (4.3.5), are quadratic functions of t. I.e. $\tilde{p}_T^i = p_0^i + tp_1^i + t^2 p_2^i$, with p_0^i, p_1^i, and p_2^i independent of t. The restriction on the body force density and lateral surface force density can be relaxed.

The model is a 2D variational problem defined on the space $H = \underset{\sim}{H}{}_D^1(\omega) \times \underset{\sim}{H}{}_D^1(\omega) \times H_D^1(\omega)$. The solution of the model is composed of five two-variable functions that can approximately describe the shell displacement arising in response to the applied loads and boundary conditions. For $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$, we define the following 2D tensors.

$$\begin{aligned} \gamma_{\alpha\beta}(\underset{\sim}{u}, w) &= \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w, \\ \rho_{\alpha\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) &= \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + \frac{1}{2}(b_\beta^\lambda u_{\alpha|\lambda} + b_\alpha^\lambda u_{\beta\lambda}) - c_{\alpha\beta} w, \\ \tau_\beta(\underset{\sim}{\theta}, \underset{\sim}{u}, w) &= b_\beta^\lambda u_\lambda + \theta_\beta + \partial_\beta w, \end{aligned} \tag{6.2.3}$$

which give the membrane strain, flexural strain, and transverse shear strain engendered by the displacement functions $(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$. The model reads: Find $(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \in H$, such that

$$\frac{1}{3}\epsilon^2 \int_{\omega} a^{\alpha\beta\lambda\gamma} \rho_{\lambda\gamma}(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \rho_{\alpha\beta}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x}$$

$$+ \int_{\omega} a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x} + \frac{5}{6}\mu \int_{\omega} a^{\alpha\beta} \tau_{\beta}(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon}) \tau_{\alpha}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x}$$

$$= \langle \boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H, \quad (6.2.4)$$

in which the forth order 2D contravariant tensor $a^{\alpha\beta\lambda\gamma}$ is the elastic tensor of the shell, defined by $a^{\alpha\beta\lambda\gamma} = 2\mu a^{\alpha\lambda} a^{\beta\gamma} + \lambda^{\star} a^{\alpha\beta} a^{\lambda\gamma}$.

The resultant loading functionals are given by

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{5}{6} \int_{\omega} p_o^{\alpha} \tau_{\alpha}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x} - \frac{\lambda}{2\mu + \lambda} \int_{\omega} p_o^3 a^{\alpha\beta} \gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x}$$

$$+ \int_{\omega} [(p_e^{\alpha} + q_a^{\alpha} - 2b_{\gamma}^{\alpha} p_o^{\gamma}) y_{\alpha} + (p_o^{\alpha}|_{\alpha} + p_e^3 + q_a^3) z]\sqrt{a}\, d\underset{\sim}{x} + \int_{\gamma_T} p_0^{\alpha} y_{\alpha}, \quad (6.2.5)$$

and

$$\langle \boldsymbol{f}_1, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \int_{\omega} [\frac{1}{3} K q_a^{\alpha} y_{\alpha} + \frac{1}{3} K q_a^3 z - (b_{\gamma}^{\alpha} p_e^{\gamma} + \frac{2}{3} H q_a^{\alpha}) \phi_{\alpha}]\sqrt{a}\, d\underset{\sim}{x}$$

$$- \frac{\lambda}{3(2\mu + \lambda)} \int_{\omega} (p_e^3 + 2H p_o^3) a^{\alpha\beta} \rho_{\alpha\beta}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}\, d\underset{\sim}{x}$$

$$+ \frac{1}{3} \int_{\gamma_T} (p_1^{\alpha} \phi_{\alpha} + p_2^{\alpha} y_{\alpha} - 2p_2^3 z). \quad (6.2.6)$$

Note that the leading term p_0^3 of the transverse component of the lateral surface force is not incorporated in the leading term of the resultant loading functional $\boldsymbol{f}_0$. The reason is the same as what we remarked for the spherical shell model, see Remark 5.3.1.

This model is a close variant of the classical Naghdi shell model. See [49], [10], [18], [11], [6], [15], etc., where the latter was cited or derived in various ways. This model is different from the generally accepted Naghdi model in three ways. First, the resultant loading functional is more involved. The noticeably different form of the leading term $\boldsymbol{f}_0$ is a consequence of our loading assumption. The classical loading functional is the leading term of the functional defined in Section 7.5. The higher order term $\epsilon^2 \boldsymbol{f}_1$ does not affect the convergence and convergence rate theorems in the relative energy norm. See Section 7.1. Second, the coefficient of the transverse shear term is 5/6 rather than the usual value 1. The third, and most significant, difference is in the expression of the flexural strain $\rho_{\alpha\beta}$. Comparing our expression

$$\rho_{\alpha\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + \frac{1}{2}(b^\lambda_\beta u_{\alpha|\lambda} + b^\lambda_\alpha u_{\beta|\lambda}) - c_{\alpha\beta} w$$

with that of Naghdi's

$$\rho^N_{\alpha\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) - \frac{1}{2}(b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha u_{\lambda|\beta}) + c_{\alpha\beta} w,$$

we see the relationship

$$\rho_{\alpha\beta} = \rho^N_{\alpha\beta} + b^\lambda_\alpha \gamma_{\lambda\beta} + b^\gamma_\beta \gamma_{\gamma\alpha}, \tag{6.2.7}$$

where $\gamma_{\alpha\beta}$ is the membrane strain defined in (6.2.3).

To establish the well posedness of the classical Naghdi model, the following equivalency was proved in [11].

$$\|\underset{\approx}{\rho}^N(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\approx}{\gamma}(\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\sim}{L}^2}$$

$$\simeq \|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1}+\|\underset{\sim}{u}\|_{\underset{\sim}{H}^1}+\|w\|_{H^1} \quad \forall\ (\underset{\sim}{\theta},\underset{\sim}{u},w)\in H,$$

from which, by using the relation (6.2.7) and the observation

$$\|\underset{\approx}{\rho}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+(1+2B)\|\underset{\approx}{\gamma}(\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\sim}{L}^2}$$

$$\gtrsim \|\underset{\approx}{\rho}^N(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\approx}{\gamma}(\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\sim}{L}^2},$$

where B is the maximum absolute value of the components of the mixed curvature tensor b^α_β over ω, the following equivalency easily follows

$$\|\underset{\approx}{\rho}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\approx}{\gamma}(\underset{\sim}{u},w)\|_{\underset{\approx}{L}^2}+\|\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\sim}{L}^2}$$

$$\simeq \|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1}+\|\underset{\sim}{u}\|_{\underset{\sim}{H}^1}+\|w\|_{H^1} \quad \forall\ (\underset{\sim}{\theta},\underset{\sim}{u},w)\in H. \quad (6.2.8)$$

Since the elastic tensor $a^{\alpha\beta\lambda\gamma}$ and the contravariant metric tensor $a^{\alpha\beta}$ are uniformly positive definite and bounded, so the bilinear form in the left hand side of the model (6.2.4) is continuous and uniformly elliptic on the space H. Therefore, we have

THEOREM 6.2.1. *If the resultant loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$ in the model (6.2.4) defines a linear continuous functional on the space H, then the model has a unique solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \in H$.*

REMARK 6.2.1. *The condition of the this existence theorem can be met, if, for example, the applied force functions satisfy the condition*

$$\tilde{p}^3_\pm \in L_2(\omega), \quad q^i_a \in L_2(\omega), \quad p^\alpha_\pm \in \underset{\sim}{H}(\mathrm{div}, \omega), \quad p^i_0, p^i_1, p^i_2 \in H^{-1/2}(\partial_T \omega). \tag{6.2.9}$$

Henceforth, we will assume that the loading functions satisfy this condition.

For brevity, the membrane, flexural, and transverse shear strains engendered by the model solution will be denoted by

$$\gamma^\epsilon_{\alpha\beta} = \gamma_{\alpha\beta}(\underset{\sim}{u}^\epsilon, w^\epsilon), \quad \rho^\epsilon_{\alpha\beta} = \rho_{\alpha\beta}(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon), \quad \tau^\epsilon_\alpha = \tau_\alpha(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon).$$

6.3 Reconstruction of the stress field and displacement field

From the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$, we can reconstruct a stress field $\boldsymbol{\sigma}$ by giving its contravariant components, and a displacement field $\boldsymbol{v}$ by giving its covariant components. The displacement field is kinematically admissible, but the stress field is not exactly statically admissible since the equilibrium equation and the lateral surface force condition on Γ_T can not be precisely satisfied. We will compute the equilibrium residual, lateral force condition residual, and the constitutive residual between the constructed stress and displacement fields, and establish an identity to express the errors of the reconstructed stress and displacement fields in terms of all these residuals so that a rigorous proof of the model convergence can be obtained by bounding these residuals.

6.3.1 The stress and displacement fields

Based on the model solution, we define the following 2D symmetric tensor-valued functions $\sigma_0^{\alpha\beta}$, $\sigma_1^{\alpha\beta}$, and a 2D vector-valued function $\sigma_0^{3\alpha}$.

$$\begin{aligned} \sigma_1^{\alpha\beta} &= a^{\alpha\beta\lambda\gamma}\rho^\epsilon_{\lambda\gamma} + \frac{\lambda}{2\mu+\lambda}(p_e^3 + 2Hp_o^3)a^{\alpha\beta}, \\ \sigma_0^{\alpha\beta} &= \frac{2}{3}H\,\epsilon^2\,\sigma_1^{\alpha\beta} + a^{\alpha\beta\lambda\gamma}\gamma^\epsilon_{\lambda\gamma} + \frac{\lambda}{2\mu+\lambda}p_o^3 a^{\alpha\beta}, \\ \sigma_0^{3\alpha} &= \frac{5}{4}(\mu a^{\alpha\beta}\tau^\epsilon_\beta - p_o^\alpha). \end{aligned} \tag{6.3.1}$$

It can be verified that, in weak sense, these tensor- and vector-valued functions satisfy the following system of differential equations and boundary condition.

$$\begin{aligned} \frac{\epsilon^2}{3}\sigma_1^{\alpha\beta}|_\beta - \frac{2}{3}\sigma_0^{3\alpha} &= \epsilon^2(b^\alpha_\gamma p_e^\gamma + \frac{2}{3}Hq_a^\alpha), \\ (\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\,d^\beta_\gamma\sigma_1^{\alpha\gamma})|_\beta - \frac{2}{3}b^\alpha_\lambda\sigma_0^{3\lambda} &= 2b^\alpha_\gamma p_o^\gamma - p_e^\alpha - (1+\frac{1}{3}K\,\epsilon^2)q_a^\alpha, \\ b_{\alpha\beta}(\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\,d^\beta_\gamma\sigma_1^{\alpha\gamma}) + \frac{2}{3}\sigma_0^{3\alpha}|_\alpha &= -p_o^\alpha|_\alpha - p_e^3 - (1+\frac{1}{3}K\,\epsilon^2)q_a^3, \end{aligned} \tag{6.3.2}$$

$$(\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\,d^\beta_\gamma\sigma_1^{\alpha\gamma})n_\beta = p_0^\alpha + \frac{1}{3}\,\epsilon^2\,p_2^\alpha, \quad \sigma_1^{\alpha\beta}n_\beta = p_1^\alpha, \quad \sigma_0^{3\beta}n_\beta = -\,\epsilon^2\,p_2^3 \ \text{ on } \gamma_T. \tag{6.3.3}$$

Indeed, by substituting (6.3.1) into (6.3.2), we will recover the model (6.2.4) in differential form. In fact, this is how the model was derived.

These functions furnish the principal part of the stress field. To complete the construction of the stress field, we need to define another 2D symmetric tensor-valued function $\sigma_2^{\alpha\beta}$ and two scalars σ_0^{33} and σ_1^{33}. The tensor-valued function $\sigma_2^{\alpha\beta}$ is required to satisfy the following equation and boundary condition

$$\begin{aligned}(\sigma_2^{\alpha\beta} - \epsilon^2\, d_\gamma^\beta \sigma_1^{\alpha\gamma})|_\beta &= -4 b_\gamma^\alpha \sigma_0^{3\gamma} - K\,\epsilon^2\, q_a^\alpha \quad \text{in} \quad \omega, \\ (\sigma_2^{\alpha\beta} - \epsilon^2\, d_\gamma^\beta \sigma_1^{\alpha\gamma}) n_\beta &= \epsilon^2\, p_2^\alpha \quad \text{on} \quad \gamma_T.\end{aligned} \tag{6.3.4}$$

This system does not uniquely determine $\sigma_2^{\alpha\beta}$. We will choose $\sigma_2^{\alpha\beta}$ to minimize its $\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)$ norm, see the next subsection. The scalars are explicitly defined by

$$\sigma_0^{33} = \frac{\epsilon^2}{2}(b_{\alpha\beta}\sigma_1^{\alpha\beta} + p_e^\alpha|_\alpha - 2Hq_a^3), \tag{6.3.5}$$

$$\begin{aligned}\sigma_1^{33} = \frac{\epsilon}{2}[b_{\alpha\beta}((\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\, d_\gamma^\beta \sigma_1^{\alpha\gamma}) + \frac{2}{3} b_{\alpha\beta}(\sigma_2^{\alpha\beta} - \epsilon^2\, d_\gamma^\beta \sigma_1^{\alpha\gamma}) \\ + p_o^\alpha|_\alpha + p_e^3 + (1 + \epsilon^2\, K) q_a^3].\end{aligned} \tag{6.3.6}$$

With all these 2D functions determined, we define the contravariant components σ^{ij} of our stress field $\boldsymbol{\sigma}$ by

$$\sigma^{\alpha\beta} = \zeta^\alpha_\lambda \zeta^\beta_\gamma [\sigma_0^{\lambda\gamma} + t\sigma_1^{\lambda\gamma} + r(t)\sigma_2^{\lambda\gamma}],$$

$$\sigma^{3\alpha} = \sigma^{\alpha 3} = \frac{1}{\rho}[p_o^\alpha + tp_e^\alpha + q(t)\sigma_0^{3\alpha}], \tag{6.3.7}$$

$$\sigma^{33} = \frac{1}{\rho}[p_o^3 + tp_e^3 + q(t)\sigma_0^{33} + s(t)\sigma_1^{33}],$$

here

$$r(t) = \frac{t^2}{\epsilon^2} - \frac{1}{3}, \quad q(t) = 1 - \frac{t^2}{\epsilon^2}, \quad s(t) = \frac{t}{\epsilon}(1 - \frac{t^2}{\epsilon^2}).$$

The stress field σ^{ij} is obviously symmetric. Following classical terminologies, we call $\sigma_0^{\alpha\beta}$ the membrane stress resultant, $\sigma_1^{\alpha\beta}$ the first membrane stress moment, and $\sigma_2^{\alpha\beta}$ the second membrane stress moment.

By the definition (4.3.1), the rescaled components $\tilde{\sigma}^{ij}$ of this stress field are

$$\tilde{\sigma}^{\alpha\beta} = \rho\zeta^\beta_\gamma [\sigma_0^{\alpha\gamma} + t\sigma_1^{\alpha\gamma} + r(t)\sigma_2^{\alpha\gamma}],$$

$$\tilde{\sigma}^{3\alpha} = \tilde{\sigma}^{\alpha 3} = p_o^\alpha + tp_e^\alpha + q(t)\sigma_0^{3\alpha}, \tag{6.3.8}$$

$$\tilde{\sigma}^{33} = p_o^3 + tp_e^3 + q(t)\sigma_0^{33} + s(t)\sigma_1^{33}.$$

By using the relation $\rho\zeta^\beta_\gamma = \delta^\beta_\gamma - td^\beta_\gamma$, the rescaled membrane stress components can be written as

$$\tilde{\sigma}^{\alpha\beta} = (\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\, d^\beta_\gamma \sigma_1^{\alpha\gamma}) + t\sigma_1^{\alpha\beta} + r(t)(\sigma_2^{\alpha\beta} - \epsilon^2\, d^\beta_\gamma \sigma_1^{\alpha\gamma}) - td^\beta_\gamma[\sigma_0^{\alpha\gamma} + r(t)\sigma_2^{\alpha\gamma}].$$

By the definition (6.3.7), we easily see that the surface force conditions (4.2.4), or equivalently, (4.3.7) on $\Gamma_\pm$ are precisely satisfied by the constructed stress field. To simplify the verification of the lateral force condition, we write the rescaled lateral surface force components as

$$\begin{aligned} \tilde{p}_T^\alpha &= p_0^\alpha + tp_1^\alpha + t^2 p_2^\alpha = p_0^\alpha + \frac{1}{3}\,\epsilon^2\, p_2^\alpha + tp_1^\alpha + r(t)\,\epsilon^2\, p_2^\alpha, \\ \tilde{p}_T^3 &= p_0^3 + tp_1^3 + t^2 p_2^3 = p_0^3 + \epsilon^2\, p_2^3 + tp_1^3 - q(t)\,\epsilon^2\, p_2^3. \end{aligned} \tag{6.3.9}$$

It can be verified that the compatibility condition of the applied surface forces on the shell edges, see Remark 5.4.1, is equivalent to

$$p_o^\alpha n_\alpha = p_0^3 + \epsilon^2\, p_2^3, \quad p_e^\alpha n_\alpha = p_1^3. \tag{6.3.10}$$

On the lateral boundary Γ_T, by using the compatibility condition (6.3.10) and the boundary conditions imposed in (6.3.3) and (6.3.4), we get the residual of the lateral surface force condition:

$$\begin{aligned} (\sigma^{\alpha j} - \sigma^{*\alpha j})n_j^* &= \frac{1}{\eta}\zeta_\gamma^\alpha(\tilde{\sigma}^{\gamma\beta}n_\beta - \tilde{p}_T^\gamma) = -\frac{t}{\eta}d_\lambda^\beta[\sigma_0^{\gamma\lambda} + r(t)\sigma_2^{\gamma\lambda}]n_\beta\zeta_\gamma^\alpha, \\ (\sigma^{3j} - \sigma^{*3j})n_j^* &= \frac{1}{\eta}(\tilde{\sigma}^{3\alpha}n_\alpha - \tilde{p}_T^3) = 0. \end{aligned} \tag{6.3.11}$$

By using the identities (4.3.8), the equations in (6.3.2), and the equation in (6.3.4), we can get the residual of the equilibrium equation:

$$\begin{aligned} \sigma^{\alpha j}\|_j + q^\alpha &= -\frac{t}{\rho}\zeta^\alpha_\gamma[d^\beta_\lambda\sigma_0^{\gamma\lambda} + r(t)d^\beta_\lambda\sigma_2^{\gamma\lambda}]|_\beta, \\ \sigma^{3j}\|_j + q^3 &= -\frac{t}{\rho}b_{\alpha\beta}[d^\beta_\lambda\sigma_0^{\gamma\lambda} + r(t)d^\beta_\lambda\sigma_2^{\gamma\lambda}]. \end{aligned} \tag{6.3.12}$$

Formally, these residuals are small. More importantly, they are explicitly expressible in terms of the two-variable functions, so they are not far beyond our control.

The displacement field $\boldsymbol{v}$ is defined by giving its rescaled components:

$$\tilde{v}_\alpha = u^\epsilon_\alpha + t\theta^\epsilon_\alpha, \quad \tilde{v}_3 = w^\epsilon + tw_1 + t^2 w_2, \tag{6.3.13}$$

in which $w_1,\ w_2 \in\ H^1_D$ are two correction functions that will be defined later. This correction does not affect the basic pattern of the shell deformation which has already been well captured by the primary displacement functions $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ given by the shell model. Obviously, $\boldsymbol{v}$ is kinematically admissible.

6.3.2 The second membrane stress moment

In the construction of the stress field, the tensor-valued function $\sigma_2^{\alpha\beta}$ was required to satisfy

$$\begin{aligned} (\sigma_2^{\alpha\beta} - \epsilon^2\, d^\beta_\gamma \sigma_1^{\alpha\gamma})|_\beta &= -4b^\alpha_\gamma\sigma_0^{3\gamma} - K\,\epsilon^2\, q^\alpha_a \ \text{ in } \omega \\ (\sigma_2^{\alpha\beta} - \epsilon^2\, d^\beta_\gamma \sigma_1^{\alpha\gamma})n_\beta &= \epsilon^2\, p_2^\alpha \ \text{ on } \ \partial_T\omega. \end{aligned} \tag{6.3.14}$$

The weak form of this problem is

$$\int_\omega \sigma_2^{\alpha\beta} \frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2} \sqrt{a} d\underset{\sim}{x} = \epsilon^2 \int_\omega \sigma_1^{\alpha\beta} \frac{d^\lambda_\beta v_{\alpha|\lambda} + d^\lambda_\alpha v_{\beta|\lambda}}{2} \sqrt{a} d\underset{\sim}{x}$$

$$+ \int_\omega (4 b^\alpha_\lambda \sigma_0^{3\lambda} + \epsilon^2 K q^\alpha_a) v_\alpha \sqrt{a} d\underset{\sim}{x} + \epsilon^2 \int_{\gamma_T} p^\alpha_2 v_\alpha \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_D. \quad (6.3.15)$$

We have

THEOREM 6.3.1. *Among all symmetric tensor-valued functions satisfying the equation (6.3.15), we can choose one such that*

$$\|\sigma_2^{\alpha\beta}\|_{\underset{\approx}{L}^2} \lesssim B \|\sigma_0^{3\alpha}\|_{\underset{\sim}{L}^2} + \epsilon^2 B \|\sigma_1^{\alpha\beta}\|_{\underset{\approx}{L}^2} + \epsilon^2 K \|q^\alpha_a\|_{\underset{\sim}{L}^2} + \epsilon^2 \|p^\alpha_2\|_{\underset{\sim}{H}^{-1/2}(\partial_T \omega)}, \quad (6.3.16)$$

where $B = \max\{|b^\alpha_\beta|\}$.

To prove this theorem, we need some lemmas. Let $\bar{\gamma}_{\alpha\beta}$ be the linear continuous operator from $\underset{\sim}{H}^1_D$ to $\underset{\approx}{L}^{\text{sym}}_2$ defined by

$$\bar{\gamma}_{\alpha\beta}(\underset{\sim}{v}) = \frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2} \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\omega).$$

We have

LEMMA 6.3.2. *The operator* $\bar{\gamma}_{\alpha\beta}$ *is injective.*

Proof. The equation $\bar{\gamma}_{\alpha\beta}(\underset{\sim}{v}) = 0$ is a system of three first order PDE's:

$$\partial_1 v_1 - \Gamma^\lambda_{11} v_\lambda = 0, \quad \partial_2 v_2 - \Gamma^\lambda_{22} v_\lambda = 0, \quad \frac{\partial_1 v_2 + \partial_2 v_1}{2} - \Gamma^\lambda_{12} v_\lambda = 0,$$

and we have the boundary condition $v_\lambda = 0$ on $\partial_D\omega$. The first two equations and the boundary condition constitute an elliptic system for the variables v_1 and v_2, see [25], with the Cauchy data imposed on part of the domain boundary. Therefore, by the unique continuation theorem of Hörmander [31], v_1 and v_2 must be identically equal to zero. □

Lemma 6.3.3. *The operator $\bar\gamma_{\alpha\beta}$ defines an isomorphism between $\underset{\sim}{H}{}^1_D$ and a closed subspace $\bar{\underset{\approx}{L}}{}_2^{\mathrm{sym}}$ of $\underset{\approx}{L}{}_2^{\mathrm{sym}}$.*

Proof. Considering the compact operator $A_2 : \underset{\sim}{H}{}^1_D \longrightarrow (L_2)^3$ defined by

$$A_2(\underset{\sim}{v}) = (\Gamma^\lambda_{11} v_\lambda, \Gamma^\lambda_{12} v_\lambda, \Gamma^\lambda_{22} v_\lambda)$$

and treating $\bar\gamma_{\alpha\beta}$ as the operator A_1 in Lemma 2.3.2, the following inequality then follows from the Korn's inequality of plane elasticity,

$$\|\underset{\sim}{v}\|_{\underset{\sim}{H}{}^1_D} \lesssim \|\bar\gamma_{\alpha\beta}(\underset{\sim}{v})\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}}.$$

Therefore, the operator $\bar\gamma_{\alpha\beta}$ has closed range. □

Proof of Theorem 6.3.1. We consider the three terms in the right hand side of (6.3.15) separately, and write $\sigma_2^{\alpha\beta} = \sigma_{2,1}^{\alpha\beta} + \sigma_{2,2}^{\alpha\beta} + \sigma_{2,3}^{\alpha\beta}$. The estimate (6.3.16) will follow from superposition. The tensor valued function $\sigma_{2,1}^{\alpha\beta}$ is required to satisfy the equation

$$\int_\omega \sigma_{2,1}^{\alpha\beta} \frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2} \sqrt{a}\, d\underset{\sim}{x} = \epsilon^2 \int_\omega \sigma_1^{\alpha\beta} \frac{d^\lambda_\beta v_{\alpha|\lambda} + d^\lambda_\alpha v_{\beta|\lambda}}{2} \sqrt{a}\, d\underset{\sim}{x}.$$

We define the linear continuous operator $D_1 : \bar{\underset{\approx}{L}}{}_2^{\text{sym}} \longrightarrow \underset{\approx}{L}{}_2^{\text{sym}}$ by

$$D_1(\frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2}) = \frac{d^\lambda_\beta v_{\alpha|\lambda} + d^\lambda_\alpha v_{\beta|\lambda}}{2},$$

which is the composition of the inverse of $\bar{\gamma}_{\alpha\beta}$ and a self explanatory operator. It is obvious that the symmetric tensor $\sigma^{\alpha\beta}_{2,1} = \epsilon^2 D_1^*(\sigma^{\alpha\beta}_1)$ satisfies the above equation. Here D_1^* is the dual of D_1.

The tensor valued function $\sigma^{\alpha\beta}_{2,2}$ is required to satisfy the equation

$$\int_\omega \sigma^{\alpha\beta}_{2,2} \frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2} \sqrt{a} d\underset{\sim}{x} = \int_\omega (4b^\alpha_\lambda \sigma^{3\lambda}_0 + \epsilon^2 K q^\alpha_a) v_\alpha \sqrt{a} d\underset{\sim}{x} \quad \forall\ \underset{\sim}{v} \in\ \underset{\sim}{H}{}^1_D.$$

We consider the operator $D_2 : \bar{\underset{\approx}{L}}{}_2^{\text{sym}} \longrightarrow \underset{\sim}{L}_2$ defined by

$$D_2(\frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2}) = \underset{\sim}{v}$$

which is the composition of the inverse of $\bar{\gamma}_{\alpha\beta}$ and the identical inclusion of $\underset{\sim}{H}{}^1_D$ in $\underset{\sim}{L}_2$. The symmetric tensor determined by $\sigma^{\alpha\beta}_{2,2} = D_2^*(4b^\alpha_\lambda \sigma^{3\lambda}_0 + \epsilon^2 K q^\alpha_a)$ satisfies this equation.

The tensor valued function $\sigma^{\alpha\beta}_{2,3}$ is required to satisfy the equation

$$\int_\omega \sigma^{\alpha\beta}_{2,3} \frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2} \sqrt{a} d\underset{\sim}{x} = \epsilon^2 \int_{\gamma_T} p^\alpha_2 v_\alpha \quad \forall\ \underset{\sim}{v} \in\ \underset{\sim}{H}{}^1_D.$$

We consider the operator $D_3 : \bar{\underset{\approx}{L}}{}_2^{\text{sym}} \longrightarrow \underset{\sim}{H}{}_{00}^{1/2}(\partial_T \omega)$ defined by

$$D_3(\frac{v_{\alpha|\beta} + v_{\beta|\alpha}}{2}) = \underset{\sim}{v}$$

which is the composition of the inverse of $\bar{\gamma}_{\alpha\beta}$ and the trace operator. The symmetric tensor determined by $\sigma_{2,3}^{\alpha\beta} = \epsilon^2 D_3^*(p_2^\alpha)$ satisfies this equation. By superposition, the solution of (6.3.15) can be chosen as

$$\sigma_2^{\alpha\beta} = \epsilon^2 D_1^*(\sigma_1^{\alpha\beta}) + D_2^*(4b_\lambda^\alpha \sigma_0^{3\lambda} + \epsilon^2 K q_a^\alpha) + \epsilon^2 D_3^*(p_2^\alpha).$$

The theorem then follows from the fact that D_1^*, D_2^*, and D_3^* are bounded operators from $\underset{\approx}{L}{}_2^{\text{sym}}$, $\underset{\sim}{L}{}_2$, and $\underset{\sim}{H}{}^{-1/2}(\partial_T \omega)$ to $\underset{\approx}{L}{}_2^{\text{sym}}$, respectively. □

Note that $\bar{\underset{\approx}{L}}{}_2^{\text{sym}}$ is a closed subspace of $\underset{\approx}{L}{}_2^{\text{sym}}$ is in consistence with the nonuniqueness of the solution of (6.3.14).

6.3.3 The integration identity

Due to the residuals of the traction boundary condition (6.3.11) and the equilibrium equation (6.3.12), we can not have the two energies principle (4.2.7) precisely hold for the constructed stress and displacement fields. However, for these fields, we have the identity (6.3.17) below, which is a substitute to the two energies principle.

THEOREM 6.3.4. *For the stress field $\boldsymbol{\sigma}$ and displacement field $\boldsymbol{v}$ defined by (6.3.7) and (6.3.13), we have the following identity*

$$\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)]$$

$$= \int_{\Omega^\epsilon} [A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})][\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})] + r, \quad (6.3.17)$$

in which

$$r = 2\int_\omega \int_{-\epsilon}^{\epsilon} t[d^\beta_\lambda \sigma_0^{\alpha\lambda} + r(t) d^\beta_\lambda \sigma_2^{\alpha\lambda}](\zeta^\gamma_\alpha v^*_\gamma - \zeta^\gamma_\alpha v_\gamma)|_\beta \sqrt{a}dtd\underset{\sim}{x}$$

$$- 2\int_\omega \int_{-\epsilon}^{\epsilon} tb_{\gamma\beta}[d^\beta_\lambda \sigma_0^{\gamma\lambda} + r(t) d^\beta_\lambda \sigma_2^{\gamma\lambda}](v^*_3 - v_3)\sqrt{a}dtd\underset{\sim}{x}, \quad (6.3.18)$$

and $\boldsymbol{v}^* = v^*_i$ and $\boldsymbol{\sigma}^* = \sigma^{*ij}$ are the displacement and stress fields on the shell determined from the 3D elasticity equations respectively.

Proof. For a stress field $\boldsymbol{\sigma} = \sigma^{ij}$ and an admissible displacement field $\boldsymbol{v}$, the following identity follows from an integration by parts over the shell Ω^ϵ.

$$\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)]$$

$$= \int_{\Omega^\epsilon} [\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})][A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})] + 2\int_{\Omega^\epsilon} (\sigma^{ij}\|_j + q^i)(v^*_i - v_i)$$

$$-2[\int_{\Gamma_+} (\sigma^{3i} - \sigma^{*3i})(v^*_i - v_i) - \int_{\Gamma_-} (\sigma^{3i} - \sigma^{*3i})(v^*_i - v_i) + \int_{\Gamma_T} (\sigma^{\alpha i} - \sigma^{*\alpha i}) n^*_\alpha (v^*_i - v_i)].$$

$$(6.3.19)$$

Substituting the displacement field $\boldsymbol{v}$ (6.3.13), the stress field $\boldsymbol{\sigma}$ (6.3.7), the residual of the lateral surface force condition (6.3.11), and the residual of the equilibrium equation (6.3.12) into the integration identity (6.3.19), we get

$$\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)]$$
$$= \int_{\Omega^\epsilon}[\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})][A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})] - 2\int_{\Omega^\epsilon}\frac{t}{\rho}b_{\alpha\beta}[d^\beta_\lambda\sigma_0^{\gamma\lambda} + r(t)d^\beta_\lambda\sigma_2^{\gamma\lambda}](v_3^* - v_3)$$
$$-2\int_{\Omega^\epsilon}\frac{t}{\rho}\zeta^\alpha_\gamma[d^\beta_\lambda\sigma_0^{\gamma\lambda} + r(t)d^\beta_\lambda\sigma_2^{\gamma\lambda}]|_\beta(v_\alpha^* - v_\alpha) + 2\int_{\Gamma_T}\frac{t}{\eta}d^\beta_\lambda[\sigma_0^{\gamma\lambda} + r(t)\sigma_2^{\gamma\lambda}]n_\beta\zeta^\alpha_\gamma(v_\alpha^* - v_\alpha).$$

(6.3.20)

The key observation here is that the last two integrals in this identity can be merged into a single term. Recalling the meanings of ρ and η, we can convert the integrals to the coordinate domain ω^ϵ and write the sum of the last two integrals as

$$-2\int_{-\epsilon}^{\epsilon}\int_\omega t[d^\beta_\lambda\sigma_0^{\alpha\lambda} + r(t)d^\beta_\lambda\sigma_2^{\alpha\lambda}]|_\beta\zeta^\gamma_\alpha(v_\gamma^* - v_\gamma)\sqrt{a}\,d\underset{\sim}{x}\,dt$$
$$+2\int_{-\epsilon}^{\epsilon}\int_{\partial_T\omega} t d^\beta_\lambda[\sigma_0^{\alpha\lambda} + r(t)\sigma_2^{\alpha\lambda}]n_\beta\zeta^\gamma_\alpha(v_\gamma^* - v_\gamma)\sqrt{a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\,ds\,dt.$$

Note that the covariant derivatives in this expression are all taken with respect to the metric on the middle surface S and $\sqrt{a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}ds = d\gamma$ is just the arc length element of γ_T. For each $t \in (-\epsilon, \epsilon)$ we can use the Green's theorem (4.1.4) on the middle surface S. The above two-term sum is equal to

$$2\int_{-\epsilon}^{\epsilon}\int_\omega t[d^\beta_\lambda\sigma_0^{\alpha\lambda} + r(t)d^\beta_\lambda\sigma_2^{\alpha\lambda}](\zeta^\gamma_\alpha v_\gamma^* - \zeta^\gamma_\alpha v_\gamma)|_\beta\sqrt{a}\,d\underset{\sim}{x}\,dt.$$

The second integral in the right hand side of the equation (6.3.20) can also be converted to the coordinate domain ω^ϵ. The desired identity then follows.

The above calculations are formal because the stress field $\boldsymbol{\sigma}$ might not be eligible for integration by parts. Note the fact that the possible singularities of the stress field $\boldsymbol{\sigma}$ arise at the lateral boundary points where the type of boundary condition changes. We can get around this difficulty by approximating $\boldsymbol{v}^* - \boldsymbol{v}$ by infinitely smooth functions with compact supports in the domain $(\partial_T\omega \cup \omega) \times [-\epsilon, \epsilon]$. Here, we assume that $\partial_T\omega = \partial\omega - \overline{\partial_D\omega}$. □

Using the fact that $r(t)$ is an even function of t and the expression (6.3.13), we can further write the expression of r as

$$
\begin{aligned}
r = 2\int_\omega \int_{-\epsilon}^{\epsilon} t[d_\lambda^\beta \sigma_0^{\alpha\lambda} + r(t) d_\lambda^\beta \sigma_2^{\alpha\lambda}](\zeta_\alpha^\gamma v_\gamma^*)|_\beta \sqrt{a}\, dt\, d\underset{\sim}{x} \\
- 2\int_\omega \int_{-\epsilon}^{\epsilon} t b_{\gamma\beta}[d_\lambda^\beta \sigma_0^{\gamma\lambda} + r(t) d_\lambda^\beta \sigma_2^{\gamma\lambda}] v_3^* \sqrt{a}\, dt\, d\underset{\sim}{x} \\
- 2\int_\omega \int_{-\epsilon}^{\epsilon} t^2[d_\lambda^\beta \sigma_0^{\alpha\lambda} + r(t) d_\lambda^\beta \sigma_2^{\alpha\lambda}] \theta_{\alpha|\beta} \sqrt{a}\, dt\, d\underset{\sim}{x} \\
+ 2\int_\omega \int_{-\epsilon}^{\epsilon} t^2 b_{\gamma\beta}[d_\lambda^\beta \sigma_0^{\gamma\lambda} + r(t) d_\lambda^\beta \sigma_2^{\gamma\lambda}] w_1 \sqrt{a}\, dt\, d\underset{\sim}{x}. \quad (6.3.21)
\end{aligned}
$$

The identity (6.3.17) expresses the energy norms of the errors of the stress field $\boldsymbol{\sigma}$ and displacement field $\boldsymbol{v}$ in terms of the constitutive residual between these two fields, and the extra term r. Note that in the expression of r, the 3D displacement $\boldsymbol{v}^*$ was involved. To bound r, in addition to knowledge of the behavior of the model solution, some bounds on the 3D displacement $\boldsymbol{v}^*$ will also be needed.

Remark 6.3.1. *If the body force density $\boldsymbol{q}$ is not constant in the transverse coordinate, the additional term*

$$2\int_{\omega^\epsilon}[\zeta^\alpha_\gamma(\tilde{q}^\gamma - \rho q^\gamma_a)(v^*_\alpha - v_\alpha) + (\tilde{q}^3 - \rho q^3_a)(v^*_3 - v_3)]$$

needs to be added to the expression (6.3.18) of r. Under the assumption

$$\|\tilde{q}^i - \rho q^i_a\|_{L_2(\omega^\epsilon)} \lesssim o(\epsilon^{3/2}), \tag{6.3.22}$$

the ensuing analyses can be carried through, and the convergence theorems can be proved in all the cases.

If the rescaled lateral surface force components $\tilde{q}^i_T$ are not quadratic polynomials in t, we can replace $\tilde{q}^\alpha_T$ by their quadratic Legendre expansions $\bar{q}^\alpha_T$, and $\tilde{q}^3_T$ by the quadratic interpolation $\bar{q}^3_T$ at the points $-\epsilon, 0, \epsilon$, see (6.3.9) for explanations, and we need to add yet another term

$$-2\int_{\partial_T\omega}\int_{-\epsilon}^{\epsilon}[(\bar{p}^\alpha_T - \tilde{p}^\alpha_T)\zeta^\gamma_\alpha(v^*_\alpha - v_\alpha) + (\bar{p}^3_T - \tilde{p}^3_T)(v^*_3 - v_3)]$$

to the expression of r. The following convergence theorems can be proved in all the cases if

$$\|\bar{p}^i_T - \tilde{p}^i_T\|_{L_2[\partial_T\omega\times(-\epsilon,\epsilon)]} \lesssim o(\epsilon^{3/2}). \tag{6.3.23}$$

6.3.4 Constitutive residual

In this subsection, we compute the constitutive residual $\varrho_{ij} = A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})$ for the stress field $\boldsymbol{\sigma}$ (6.3.7) and the displacement field $\boldsymbol{v}$ (6.3.13) constructed previously. First, by using Lemma 4.3.2 and the definition (6.2.3), we can compute the strain tensor engendered by the displacement $\boldsymbol{v}$. The result is

$$\begin{aligned}
\chi_{\alpha\beta}(\boldsymbol{v}) &= \gamma^{\epsilon}_{\alpha\beta} + t\rho^{\epsilon}_{\alpha\beta} - t(b^{\lambda}_{\alpha}\gamma^{\epsilon}_{\lambda\beta} + b^{\lambda}_{\beta}\gamma^{\epsilon}_{\lambda\alpha}) + (tc_{\alpha\beta} - b_{\alpha\beta})(tw_1 + t^2 w_2) \\
&\quad -\frac{1}{2}t^2(b^{\gamma}_{\alpha}\theta^{\epsilon}_{\gamma|\beta} + b^{\gamma}_{\beta}\theta^{\epsilon}_{\gamma|\alpha}), \\
\chi_{\alpha 3}(\boldsymbol{v}) &= \chi_{3\alpha}(\boldsymbol{v}) = \frac{1}{2}\tau^{\epsilon}_{\alpha} + \frac{1}{2}(t\partial_{\alpha}w_1 + t^2\partial_{\alpha}w_2), \quad \chi_{33}(\boldsymbol{v}) = w_1 + 2tw_2. \qquad (6.3.24)
\end{aligned}$$

Next, by using the definition of the 3D compliance tensor A_{ijkl}, the relation (4.1.5), the formulae (6.3.7), the definition (6.3.1), and the identities

$$\frac{1}{2\mu}(a_{\alpha\lambda}b_{\beta\gamma}a^{\lambda\gamma} + b_{\alpha\lambda}a_{\beta\gamma}a^{\lambda\gamma} - \frac{2\lambda}{2\mu+3\lambda}b_{\alpha\beta}a_{\lambda\gamma}a^{\lambda\gamma}) = \frac{2(2\mu+\lambda)}{2\mu(2\mu+3\lambda)}b_{\alpha\beta}$$

and

$$\frac{1}{2\mu}(a_{\alpha\lambda}b_{\beta\gamma}a^{\lambda\gamma\delta\rho}\tau_{\delta\rho} + b_{\alpha\lambda}a_{\beta\gamma}a^{\lambda\gamma\delta\rho}\tau_{\delta\rho} - \frac{2\lambda}{2\mu+3\lambda}b_{\alpha\beta}a_{\lambda\gamma}a^{\lambda\gamma\delta\rho}\tau_{\delta\rho}) = b^{\lambda}_{\alpha}\tau_{\lambda\beta} + b^{\lambda}_{\beta}\tau_{\lambda\alpha}$$

for any symmetric tensor $\tau_{\alpha\beta}$, after a lengthy calculation, we get the following expressions for $A_{ijkl}\sigma^{kl}$:

$$A_{\alpha\beta kl}\sigma^{kl} = \gamma^{\epsilon}_{\alpha\beta} + t\rho^{\epsilon}_{\alpha\beta} - t(b^{\lambda}_{\alpha}\gamma^{\epsilon}_{\lambda\beta} + b^{\lambda}_{\beta}\gamma^{\epsilon}_{\lambda\alpha})$$

$$+\frac{\lambda}{2\mu(2\mu+3\lambda)}[a_{\alpha\beta}(p_o^3+tp_e^3)+2t(Ha_{\alpha\beta}-b_{\alpha\beta})p_o^3]$$
$$+\frac{2}{3}H^2\,\epsilon^2[\rho^\epsilon_{\alpha\beta}+\frac{\lambda}{2\mu(2\mu+3\lambda)}a_{\alpha\beta}(p_e^3+2Hp_o^3)]$$
$$-(t^2+\frac{2}{3}Ht\,\epsilon^2)[b^\lambda_\alpha\rho^\epsilon_{\lambda\beta}+b^\lambda_\beta\rho^\epsilon_{\lambda\alpha}-\frac{2\lambda}{2\mu(2\mu+3\lambda)}b_{\alpha\beta}(p_e^3+2Hp_o^3)]$$
$$+\frac{t^2}{2\mu}(b_{\alpha\lambda}b_{\beta\gamma}-\frac{\lambda}{2\mu+3\lambda}b_{\alpha\beta}a_{\lambda\gamma})[\sigma_0^{\lambda\gamma}+t\sigma_1^{\lambda\gamma}+r(t)\sigma_2^{\lambda\gamma}]$$
$$+\frac{r(t)}{2\mu}[(a_{\alpha\lambda}-tb_{\alpha\lambda})(a_{\beta\gamma}-tb_{\beta\gamma})\sigma_2^{\lambda\gamma}-\frac{\lambda}{2\mu+3\lambda}g_{\alpha\beta}a_{\lambda\gamma}\sigma_2^{\lambda\gamma}]$$
$$-\frac{\lambda}{2\mu(2\mu+3\lambda)}g_{\alpha\beta}[p_o^3+tp_e^3+q(t)\sigma_0^{33}+s(t)\sigma_1^{33}], \tag{6.3.25}$$

$$A_{3\alpha kl}\sigma^{kl} = \frac{1}{2\mu}a_{\alpha\gamma}[p_o^\gamma+\sigma_0^{3\gamma}q(t)]+\frac{t}{2\mu}\{a_{\alpha\gamma}p_e^\gamma$$
$$+(tc_{\alpha\gamma}-2b_{\alpha\gamma}+\frac{2H-tK}{\rho}g_{\alpha\gamma})[p_o^\gamma+tp_e^\gamma+q(t)\sigma_0^{3\gamma}]\}, \tag{6.3.26}$$

$$A_{33kl}\sigma^{kl} = \frac{1}{E}(p_o^3-\nu a_{\alpha\beta}\sigma_0^{\alpha\beta})+t\frac{1}{E}(p_e^3-\nu a_{\alpha\beta}\sigma_1^{\alpha\beta})-\frac{\nu}{E}r(t)a_{\alpha\beta}\sigma_2^{\alpha\beta}$$
$$+\frac{1}{E}\{q(t)\sigma_0^{33}+s(t)\sigma_1^{33}$$
$$+t\frac{2H-tK}{\rho}[p_o^3+tp_e^3+q(t)\sigma_0^{33}+s(t)\sigma_1^{33}]\}, \tag{6.3.27}$$

where

$$E=\frac{\mu(2\mu+3\lambda)}{\mu+\lambda}\quad\text{and}\quad\nu=\frac{\lambda}{2(\mu+\lambda)}$$

are the Young's modulus and Poisson ratio of the elastic material comprising the shell.

Combining (6.3.1), (6.3.24), (6.3.25), (6.3.26), and (6.3.27), after some calculations, we get the explicit expression of the constitutive residual:

$$\varrho_{\alpha\beta}=\frac{2}{3}H^2\,\epsilon^2\,\rho^\epsilon_{\alpha\beta}-(t^2+\frac{2}{3}Ht\,\epsilon^2)(b^\lambda_\alpha\rho^\epsilon_{\lambda\beta}+b^\lambda_\beta\rho^\epsilon_{\lambda\alpha})+\frac{1}{2}t^2(b^\lambda_\alpha\theta^\epsilon_{\lambda|\beta}+b^\lambda_\beta\theta^\epsilon_{\lambda|\alpha})$$

$$+\frac{t^2}{2\mu}(b_{\alpha\lambda}b_{\beta\gamma}-\frac{\lambda}{2\mu+3\lambda}b_{\alpha\beta}a_{\lambda\gamma})[\sigma_0^{\lambda\gamma}+t\sigma_1^{\lambda\gamma}+r(t)\sigma_2^{\lambda\gamma}]$$
$$-\frac{\lambda}{2\mu(2\mu+3\lambda)}g_{\alpha\beta}[q(t)\sigma_0^{33}+s(t)\sigma_1^{33}]$$
$$+\frac{r(t)}{2\mu}[(a_{\alpha\lambda}-tb_{\alpha\lambda})(a_{\beta\gamma}-tb_{\beta\gamma})\sigma_2^{\lambda\gamma}-\frac{\lambda}{2\mu+3\lambda}g_{\alpha\beta}a_{\lambda\gamma}\sigma_2^{\lambda\gamma}]$$
$$+\frac{\lambda}{2\mu(2\mu+3\lambda)}[(2tH+\frac{4}{3}H^3\,\epsilon^2)a_{\alpha\beta}+4H(t^2+\frac{2}{3}Ht\,\epsilon^2)b_{\alpha\beta}-t^2c_{\alpha\beta}]p_o^3$$
$$+\frac{\lambda}{2\mu(2\mu+3\lambda)}[\frac{2}{3}H^2\,\epsilon^2\,a_{\alpha\beta}+(4t^2+\frac{4}{3}Ht\,\epsilon^2)b_{\alpha\beta}-t^3c_{\alpha\beta}]p_e^3$$
$$+(b_{\alpha\beta}-tc_{\alpha\beta})(tw_1+t^2w_2), \tag{6.3.28}$$

$$\varrho_{3\alpha}=\frac{1}{2\mu}a_{\alpha\lambda}\sigma_0^{3\lambda}[q(t)-\frac{4}{5}]-\frac{t}{2}\partial_\alpha w_1-\frac{t^2}{2}\partial_\alpha w_2$$
$$+\frac{t}{2\mu}\{a_{\alpha\lambda}p_e^\lambda+(tc_{\alpha\lambda}-b_{\alpha\lambda}+\frac{2H-tK}{\rho}g_{\alpha\lambda})[p_o^\lambda+tp_e^\lambda+q(t)\sigma_0^{3\lambda}]\}, \tag{6.3.29}$$

$$\varrho_{33}=[\frac{1}{E}(p_o^3-\nu a_{\alpha\beta}\sigma_0^{\alpha\beta})-w_1]+t[\frac{1}{E}(p_e^3-\nu a_{\alpha\beta}\sigma_1^{\alpha\beta})-2w_2]$$
$$+\frac{1}{E}[q(t)\sigma_0^{33}+s(t)\sigma_1^{33}-\nu a_{\alpha\beta}r(t)\sigma_2^{\alpha\beta}]$$
$$+\frac{t}{E}\frac{2H-tK}{\rho}[p_o^3+tp_e^3+q(t)\sigma_0^{33}+s(t)\sigma_1^{33}]. \tag{6.3.30}$$

REMARK 6.3.2. *If we had not defined the flexural strain $\rho_{\alpha\beta}$ different from that of Naghdi's ($\rho^N_{\alpha\beta}$), there would be an additional term $t(b^\lambda_\alpha\gamma_{\lambda\beta}+b^\lambda_\beta\gamma_{\lambda\alpha})$ in the residual $\varrho_{\alpha\beta}$. Our variant does make the constitutive residual smaller, at least formally.*

Based on their involvements in (6.3.29) and (6.3.30), the two correction functions w_1 and w_2 will be chosen to make

$$\frac{1}{E}(p_o^3-\nu a_{\alpha\beta}\sigma_0^{\alpha\beta})-w_1=-\frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}\gamma^\epsilon_{\alpha\beta}+\frac{1}{2\mu+\lambda}p_o^3-\frac{\nu}{E}\frac{3}{2}H\,\epsilon^2\,a_{\alpha\beta}\sigma_1^{\alpha\beta}-w_1 \tag{6.3.31}$$

and

$$\frac{1}{E}(p_e^3 - \nu a_{\alpha\beta}\sigma_1^{\alpha\beta}) - 2w_2$$
$$= -\frac{\lambda}{2\mu+\lambda}a^{\alpha\beta}\rho_{\alpha\beta}^{\epsilon} + \frac{1}{2\mu+\lambda}p_e^3 - \frac{2H\lambda^2}{\mu(2\mu+\lambda)(2\mu+3\lambda)}p_o^3 - 2w_2 \quad (6.3.32)$$

small in the $L_2(\omega)$ norm. At the same time, their $H^1(\omega)$ norms must be kept under control.

These formulae make it possible to prove the the model convergence. A rigorous justification of the model requires a great deal of information about the behavior of the model solution, and we must consider the relative energy norm. In addition to the upper bound on the residuals, we also need a lower bound on the energy contained in the 2D model solution. Since the 3D solution $\boldsymbol{v}^*$ was involved in the extra term r in the identity (6.3.17), we also need to bound the 3D solution. To this end, we need a Korn-type inequality on thin shells.

6.3.5 A Korn-type inequality on three-dimensional thin shells

In this subsection, we establish an inequality to bound the term r in the integration identity (6.3.17). With this inequality, we will be able to show that the extra term r, which is due to the the residuals of equilibrium equation and lateral surface force condition of our almost admissible stress field, do not affect the convergence of the model solution toward the 3D solution in the cases of flexural shells and totally clamped elliptic shells. For all the other membrane–shear shells, this inequality will be used to prove the convergence theorem under some other assumptions on the loading functions.

It is well known that Korn's inequality, which bounds the H^1 norm of a displacement field by its strain energy norm, contains a constant depending on the shape and size of the elastic body. On a thin shell, the H^1 norm of a displacement field can not be bounded by its strain energy norm uniformly with respect to the shell thickness. The following ϵ-dependent inequality (6.3.35) of Korn-type was established in [18] and [22], see also [32] and [1] for similar results.

Let $\omega^1 = \omega \times (-1, 1)$ be the scaled coordinate domain, and $\partial_D \omega^1 = \partial_D \omega \times [-1, 1]$ be the part of the scaled clamping lateral boundary. For any $\boldsymbol{v} \in \boldsymbol{H}^1_D(\omega^1)$, we define a displacement field $\boldsymbol{v}^\epsilon$ on Ω^ϵ by

$$\boldsymbol{v}^\epsilon(\underset{\sim}{x}, t) = \boldsymbol{v}(\underset{\sim}{x}, \frac{t}{\epsilon}) \ \ \forall \ \underset{\sim}{x} \in \omega, \ t \in (-\epsilon, \epsilon). \tag{6.3.33}$$

We define the scaled strain tensor for the vector field $\boldsymbol{v}$ by

$$\chi^\epsilon_{ij}(\boldsymbol{v}) = \chi_{ij}(\boldsymbol{v}^\epsilon). \tag{6.3.34}$$

There exists an $\epsilon_0 > 0$, such that when $\epsilon \le \epsilon_0$ the inequality

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1_D(\omega^1)} \lesssim \epsilon^{-2} \sum_{i,j=1}^{3} \|\chi^\epsilon_{ij}(\boldsymbol{v})\|^2_{L_2(\omega^1)} \tag{6.3.35}$$

uniformly holds for all ϵ and $\boldsymbol{v} \in \boldsymbol{H}^1_D(\omega^1)$.

From this inequality, we immediately have

Theorem 6.3.5. *There exists a constant* $\epsilon_0 > 0$ *such that, for all* $\epsilon \leq \epsilon^0$ *and any* $\boldsymbol{v} = v_i \in \boldsymbol{H}^1_D(\omega^\epsilon)$, *we have*

$$\sum_{\alpha=1}^{2} \|v_\alpha\|^2_{H^1(\omega)\times L_2(-\epsilon,\epsilon)} + \|v_3\|^2_{L_2(\omega^\epsilon)} \lesssim \epsilon^{-2} \sum_{i,j=1}^{3} \|\chi_{ij}(\boldsymbol{v})\|^2_{L_2(\omega^\epsilon)}.$$

6.4 Classification

The shell model (6.2.4) is an ϵ-dependent variational problem whose solution can behave dramatically different in different circumstances. To get accurate *a priori* estimates, the problem must be classified. By making some assumptions on the applied forces, we can fit the shell model into the abstract problem (3.2.2) of Chapter 3, and accordingly classify the problem.

6.4.1 Assumptions on the loading functions

We assume all the loading functions explicitly involved in the model, namely, the odd and weighted even parts p^i_o and p^i_e of the applied surface forces, the coefficients p^i_0, p^i_1, and p^i_2 of the rescaled lateral surface force, and the components q^i_a of the body force, are independent of ϵ.

Roughly speaking, the convergence theory established under this assumption has the physical meaning that when the model is applied to a realistic shell, no matter how the shell is loaded, the thinner the shell the better the results the model provides.

6.4.2 Classification

To use the results of Chapter 3, we introduce the following spaces and operators. As above, $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$ with the usual product norm. We let $U = \underset{\approx}{L}_2^{\mathrm{sym}}(\omega)$ with the equivalent inner product

$$(\underset{\approx}{\rho}^1, \underset{\approx}{\rho}^2)_U = \frac{1}{3}\int_\omega a^{\alpha\beta\lambda\gamma}\rho^1_{\lambda\gamma}\rho^2_{\alpha\beta}\sqrt{a}d\underset{\sim}{x} \;\; \forall\; \underset{\approx}{\rho}^1, \underset{\approx}{\rho}^2 \in U,$$

and define $A : H \to U$, the flexural strain operator, by

$$A(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \underset{\approx}{\rho}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \;\; \forall\; (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H.$$

We also define $B : H \to \underset{\approx}{L}_2^{\mathrm{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, combining the membrane and shear strain operators, by

$$B(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = [\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)] \;\; \forall\; (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H.$$

We introduce the space $W = B(H) \subset \underset{\approx}{L}_2^{\mathrm{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, in which the norm is defined by

$$\|[\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)]\|_W = \inf_{[\underset{\approx}{\gamma}(\bar{\underset{\sim}{u}}, \bar{w}), \underset{\sim}{\tau}(\bar{\underset{\sim}{\theta}}, \bar{\underset{\sim}{u}}, \bar{w})] = [\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)]} \|(\bar{\underset{\sim}{\theta}}, \bar{\underset{\sim}{u}}, \bar{w})\|_H$$

$$\forall\; [\underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)] \in W.$$

Equipped with this norm, W is a Hilbert space isomorphic to $H/\ker(B)$. The operator B is, of course, an onto mapping from H to W.

The space V is defined as the closure of W in $\underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, with the inner product

$$((\underset{\approx}{\gamma}^1, \underset{\sim}{\tau}^1), (\underset{\approx}{\gamma}^2, \underset{\sim}{\tau}^2))_V = \int_\omega a^{\alpha\beta\lambda\gamma} \gamma^1_{\lambda\gamma} \gamma^2_{\alpha\beta} \sqrt{a} d\underset{\sim}{x} + \frac{5}{6}\mu \int_\omega a^{\alpha\beta} \tau^1_\beta \tau^2_\alpha \sqrt{a} d\underset{\sim}{x},$$

which is equivalent to the inner product of $\underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$.

The range of the operator B then is dense in V, as was required by the abstract theory of Chapter 3. The space V actually is equal to the product of V_0, the closure of the range of $\underset{\approx}{\gamma}$ in $\underset{\approx}{L}_2^{\text{sym}}(\omega)$, and the closure of the range of $\underset{\sim}{\tau}$ in $\underset{\sim}{L}_2(\omega)$. The latter, since the range of $\underset{\sim}{\tau}$ is dense in $\underset{\sim}{L}_2(\omega)$, is just equal to $\underset{\sim}{L}_2(\omega)$, so we have the factorization

$$V = V_0 \times \underset{\sim}{L}_2(\omega). \tag{6.4.1}$$

From the definitions of the membrane, flexural, and shear strains (6.2.3), we easily see that A and B are continuous operators. The equivalency (6.2.8) guaranteed the condition (3.2.1).

Remark 6.4.1. *It should be noted that, in contrast to the fact that V is a product space, the space W can not be viewed as a product space generally. If the shell is flat, the membrane strain will be separated from the flexural and shear strains. The model is split to the Reissner–Mindlin plate stretching model and bending model. When the flat shell (plate) is totally clamped, the space W can be identified as [a closed subspace of $\underset{\approx}{L}_2^{\text{sym}}$] $\times\ \mathring{\underset{\sim}{H}}(\text{rot})$, see [8], [13] and [35]. For the plane strain cylindrical shell problems, the operator B has* *closed range, and so W is simply equal to V. For general shells, the characterization of W in the Sobolev sense is either unclear or impossible.*

Under the loading assumption, in the resultant loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_1$ of the model (6.2.4), both $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$, see (6.2.5) and (6.2.6), are independent of ϵ, so the loading functional is rightfully in the form of the right hand side of the abstract problem (3.2.2) of Chapter 3. According to the classification of Section 3.5, if $\boldsymbol{f}_0|_{\ker B} \neq 0$, the shell problem is called a flexural shell. If $\boldsymbol{f}_0|_{\ker B} = 0$, then, since B is surjective from H to W, by the closed range theorem, there exists a unique $\zeta_*^0 \in W^*$ such that

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \rangle = \langle \zeta_*^0, B(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \rangle \quad \forall\ (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H.$$

If $\zeta_*^0 \in V^*$, the shell problem is called a membrane–shear shell. If $\boldsymbol{f}_0|_{\ker B} = 0$, but ζ_*^0 is not in V^*, the shell model is not justified.

The kernel space $\ker B$, according to the definition of the operator B, is composed of admissible displacement fields of the form $(u_\alpha + t\theta_\alpha)\boldsymbol{a}^\alpha + w\boldsymbol{a}^3$, from which the engendered membrane strain $\underset{\approx}{\gamma}(\underset{\sim}{u}, w)$ and the transverse shear strain $\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$ vanish. The displacement in this space is pure flexural. In this kind of deformation, the intrinsic metric of the middle surface does not change infinitesimally, and there is no transverse shear strain. The condition $\boldsymbol{f}_0|_{\ker B} \neq 0$ means that the applied forces do bring about the pure flexural deformation. Thus the name flexural shell.

In the case of membrane–shear shells, the membrane energy and transverse shear energy together dominate the strain energy. It seems that there is no way to distinguish the contributions to the total energy from the membrane and the transverse shear strains. This is the reason why we call the shells in the second category the membran–shear shells.

Flexural shells, of course, require $\ker B \neq 0$ (pure flexural deformation is not inhibited). Membrane–shear shells include two different kinds, namely, $\ker B = 0$ (pure flexural is inhibited, henceforth, shells of this kind will be called stiff shells) and $\ker B \neq 0$ but $\boldsymbol{f}_0|_{\ker B} = 0$ (pure flexural is not inhibited, but the loading function does not make the pure flexural happen). A typical example of this second kind membrane–shear shells is plate stretching.

6.5 Flexural shells

We prove the convergence of the 2D model solution toward the 3D solution in the relative energy norm for flexural shells as classified in the last section. The convergence can be proved without any extra assumption. Under some regularity assumption on the solution of the limiting flexural model (6.5.3), convergence rate (as a power of ϵ) will be established.

First, we resolve the term r in the identity (6.3.17). From (6.3.18), we see that

$$|r| \lesssim \epsilon[\|\underset{\approx}{\sigma}_0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)} + \|\underset{\approx}{\sigma}_2\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}][\|\underset{\sim}{v}^* - \underset{\sim}{v}\|_{\underset{\sim}{H}^1(\omega)\times \underset{\sim}{L}_2(-\epsilon,\epsilon)} + \|v_3^* - v_3\|_{L_2(\omega^\epsilon)}].$$

We will show that $\|\underset{\approx}{\sigma}_0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}$ and $\|\underset{\approx}{\sigma}_2\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}$ are so small in the case of flexural shells that we can totally give the factor ϵ to the second half of the above right hand side. Using Theorem 6.3.5 and Cauchy's inequality to the identity (6.3.17), we get

$$\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)]$$

$$\lesssim \int_{\Omega^\epsilon} [A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})][\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})] + \|\underset{\approx}{\sigma}_0\|^2_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)} + \|\underset{\approx}{\sigma}_2\|^2_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}. \tag{6.5.1}$$

6.5.1 Asymptotic behavior of the model solution

As we have seen in Chapter 3, if the shell is flexural, the model solution blows up at the rate of $O(\epsilon^{-2})$. To get more accurate estimates, we need to scale the loading functions by assuming

$$p_o^i = \epsilon^2 P_o^i, \quad p_e^i = \epsilon^2 P_e^i, \quad q_a^i = \epsilon^2 Q_a^i, \quad p_0^i = \epsilon^2 P_0^i, \quad p_1^i = \epsilon^2 P_1^i, \quad p_2^i = \epsilon^2 P_2^i, \tag{6.5.2}$$

with $P_o^i, P_e^i, Q_a^i, P_0^i, P_1^i$, and P_2^i independent of ϵ. Therefore, $\boldsymbol{F}_0 = \epsilon^{-2}\boldsymbol{f}_0$ is a functional independent of ϵ. Since we will consider the relative energy norm, this assumption is not a requirement on the applied loads. It is just a technique to ease the analysis.

Under this scaling, the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ converges to the solution $(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)$ of the ϵ-independent limiting problem:

$$(\underset{\approx}{\rho}(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0), \underset{\approx}{\rho}(\underset{\sim}{\phi}, \underset{\sim}{y}, z))_U + \langle \xi^0, [\underset{\approx}{\gamma}(\underset{\sim}{y}, z), \underset{\sim}{\tau}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)]\rangle = \langle \boldsymbol{F}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle,$$

$$\langle \eta, [\underset{\approx}{\gamma}(\underset{\sim}{u}^0, w^0), \underset{\sim}{\tau}(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)]\rangle = 0, \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H, \ \ \forall\ \eta \in W^*, \tag{6.5.3}$$

$$(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0) \in H, \ \xi^0 \in W^*.$$

This problem has a unique solution $(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0) \in \ker B$, $\xi^0 \in W^*$. It is important to note that $\underset{\approx}{\rho}(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0) \neq 0$. Otherwise, $(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0) = 0$, which is contradicted to the flexural assumption $\boldsymbol{f}_0|_{\ker B} \neq 0$. From (3.3.5) of Chapter 3, we have

$$\|(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)\|_H + \|\xi^0\|_{W^*} \simeq \|\boldsymbol{F}_0\|_{H^*}.$$

The equation (6.5.3) is the limiting flexural shell model. This equation and its solution provide indispensable supports to the ensuing analysis. For brevity, we denote $\rho^0_{\alpha\beta} = \rho_{\alpha\beta}(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)$.

Without any assumption on the regularity of the Lagrange multiplier $\xi^0 \in W^*$ defined in the limiting problem (6.5.3), according to Theorem 3.3.3 and (3.4.5), we have the strong convergence

$$\|\underset{\approx}{\rho}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \epsilon^{-1}\|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \epsilon^{-1}\|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{6.5.4}$$

If we assume more regularity on ξ^0, say,

$$\xi^0 \in [V^*, W^*]_{1-\theta,q} \tag{6.5.5}$$

for some $\theta \in (0,1)$ and $q \in [1,\infty]$ or $\theta \in [0,1]$ and $q \in (1,\infty)$, by Theorem 3.3.2 and (3.4.4), we have

$$\|\underset{\approx}{\rho}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \epsilon^{-1}\|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \epsilon^{-1}\|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim K(\epsilon, \xi^0, [W^*, V^*]) \lesssim \epsilon^\theta. \tag{6.5.6}$$

Recall that the K-functional on the Hilbert couple $[W^*, V^*]$, see [9], is defined as

$$K(\epsilon, \xi^0, [W^*, V^*]) \simeq \|\xi^0\|_{W^* + \epsilon V^*} \simeq \inf_{\xi^0 = \xi_1^0 + \xi_2^0} (\|\xi_1^0\|_{W^*} + \epsilon \, \|\xi_2^0\|_{V^*}). \tag{6.5.7}$$

Based on the requirements imposed on the correction functions w_1 and w_2, and recalling the expressions (6.3.31) and (6.3.32) which need to be small, we define

$$w_1 = 0 \tag{6.5.8}$$

and define w_2 as the solution of

$$\begin{gathered} \epsilon^2 (\nabla w_2, \nabla v)_{\underset{\sim}{L}_2(\omega)} + (w_2, v)_{L_2(\omega)} = -\frac{\lambda}{2(2\mu + \lambda)} (a^{\alpha\beta} \rho^0_{\alpha\beta}, v)_{L_2(\omega)}, \\ w_2 \in H^1_D(\omega), \quad \forall \, v \in H^1_D(\omega). \end{gathered} \tag{6.5.9}$$

The right hand side of the equation (6.5.9) is not a trivial extension of its analogue in the Reissner–Mindlin plate theory developed in [2], according to which, $\rho^\epsilon_{\alpha\beta}$, rather than $\rho^0_{\alpha\beta}$, would have been used. We make this choice not only because of lack of regularity of the ϵ dependent model solution, this choice of the correction functions is also sufficient for us to prove the convergence and determine the convergence rate in the next two subsections. The physical meaning of (6.5.8) is that, in the flexural dominating deformation, the change of the shell thickness is negligible. In contrast, the relative motion of the location of the middle surface is significant. For example, if locally, the shell were bent down, the middle point would move toward the upper surface and *vise versa.* The existence of such correction functions such that the convergence can be proved is a sufficient justification of the model. Note that the correction does not affect the middle surface deformation, which has already been well captured by the model solution. In the forthcoming analysis of membrane–shear shells, we will choose the opposite, $w_2 = 0$.

Since the solution of the limiting problem (6.5.3) always guarantees that $\underset{\approx}{\rho}^0 \in \underset{\approx}{L}{}_2^{\text{sym}}(\omega)$, we have $a^{\alpha\beta}\rho^0_{\alpha\beta} \in L_2(\omega)$. By (3.3.38) in Theorem 3.3.6, we have

$$\epsilon\,\|w_2\|_{H^1(\omega)} + \| - w_2 - \frac{\lambda}{2(2\mu+\lambda)} a^{\alpha\beta}\rho^0_{\alpha\beta}\|_{L_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{6.5.10}$$

If we assume

$$a^{\alpha\beta}\rho^0_{\alpha\beta} \in [H^1_D(\omega), L_2(\omega)]_{1-\theta,p} \tag{6.5.11}$$

for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or $\theta \in [0,1]$ and $p \in (1,\infty)$, by (3.3.36) in Theorem 3.3.6, we have

$$\epsilon\,\|w_2\|_{H^1(\omega)} + \| - w_2 - \frac{\lambda}{2(2\mu+\lambda)} a^{\alpha\beta}\rho^0_{\alpha\beta}\|_{L_2(\omega)} \lesssim K(\epsilon, a^{\alpha\beta}\rho^0_{\alpha\beta}, [L_2(\omega), H^1_D(\omega)]) \lesssim \epsilon^{\theta}\,. \tag{6.5.12}$$

Remark 6.5.1. *Both the assumptions (6.5.5) and (6.5.11) are requirements on regularity of the solution of the ϵ independent limiting problem (6.5.3), which are indirect requirements on the shell data. The explicit dependence of the indices on these data needs more analysis. The value of the index θ in (6.5.5) may be different from that in (6.5.11). We choose the least one so that both the estimates (6.5.6) and (6.5.12) hold simultaneously.*

The asymptotic behavior of the model solution described by (6.5.4) and (6.5.6) together with the equivalency (6.2.8) tell us that under the scaling of the loading functions (6.5.2), the H^1 norm of the model solution is uniformly bounded:

$$\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} \lesssim 1, \quad \|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} \lesssim 1, \quad \|w^\epsilon\|_{H^1(\omega)} \lesssim 1,$$

while the following estimate (6.5.18) shows that the strain energy engendered by the this displacement is only of order $O(\epsilon^3)$. This is the magnitude of strain energy that flexural shells could sustain without collapsing.

6.5.2 Convergence theorems

As in Chapter 5, we denote the energy norms of a stress field $\boldsymbol{\sigma}$ and a strain field $\boldsymbol{\chi}$ on the shell Ω^ϵ by $\|\boldsymbol{\sigma}\|_{E^\epsilon}$ and $\|\boldsymbol{\chi}\|_{E^\epsilon}$, which are equivalent to the sums of the $L_2(\omega^\epsilon)$ norms of the tensor components.

Without making any assumption further than (6.2.9), it can be proved that the model solution converges to the 3D solution in the relative energy norm. We have

THEOREM 6.5.1. *Let $\boldsymbol{v}^*$ and $\boldsymbol{\sigma}^*$ be the solution of the 3D shell problem, $\boldsymbol{v}$ the displacement field defined by the solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ of the model (6.2.4) together with the correction functions w_1 and w_2 defined in (6.5.8) and (6.5.9) through the formulae (6.3.13), and $\boldsymbol{\sigma}$ the stress field defined by (6.3.7). We have the convergence*

$$\lim_{\epsilon \to 0} \frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0. \qquad (6.5.13)$$

If the solution $(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)$ and ξ^0 of the ϵ-independent limiting problem (6.5.3) satisfies the condition

$$\xi^0 \in [V^*, W^*]_{1-\theta,q}, \qquad a^{\alpha\beta}\rho^0_{\alpha\beta} \in [H^1_D(\omega), L_2(\omega)]_{1-\theta,p} \tag{6.5.14}$$

for some $\theta \in (0,1)$ and $p,\ q \in [1,\infty]$ or $\theta \in [0,1]$ and $p,\ q \in (1,\infty)$, we have the two estimates (6.5.6) and (6.5.12) hold simultaneously, and we have

THEOREM 6.5.2. *If the regularity condition (6.5.14) is satisfied for some θ, we have the convergence rate*

$$\frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^\theta\,. \tag{6.5.15}$$

We give the proof of Theorem 6.5.2. The proof of Theorem 6.5.1 is similar.

Proof. The proof is based on the inequality (6.5.1), the above two estimates (6.5.6) and (6.5.12), the inequality (6.3.16) to bound $\sigma_2^{\alpha\beta}$, the expressions (6.3.28), (6.3.29) and (6.3.30) for the constitutive residual ϱ_{ij}, and the scaling on the loads (6.5.2). In the proof, the norm $\|\cdot\|_{L_2(\omega^\epsilon)}$ will be simply denoted by $\|\cdot\|$. Any function defined on ω will be viewed as a function, constant in t, defined on ω^ϵ.

First, we establish the lower bound for $\|\boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon}$. By the estimate (6.5.6), we have

$$\|\underset{\approx}{\rho}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^\theta, \quad \|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^{1+\theta}, \quad \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^{1+\theta}, \tag{6.5.16}$$

so

$$\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \simeq \|\underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \simeq 1. \tag{6.5.17}$$

From the equivalence (6.2.8), we see

$$\|(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon})\|_{\underset{\sim}{H}^1(\omega)\times\underset{\sim}{H}^1(\omega)\times H^1(\omega)} \simeq \|\underset{\approx}{\varrho}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \simeq 1.$$

The convergence (6.5.12) shows that $\|w_2\|_{L_2(\omega)} \simeq \|a^{\alpha\beta}\rho^0_{\alpha\beta}\|_{L_2(\omega)} \lesssim 1$. Recalling the expression (6.3.24)

$$\chi_{\alpha\beta}(\boldsymbol{v}) = \gamma^{\epsilon}_{\alpha\beta} + t\rho^{\epsilon}_{\alpha\beta} - t(b^{\lambda}_{\alpha}\gamma^{\epsilon}_{\lambda\beta} + b^{\lambda}_{\beta}\gamma^{\epsilon}_{\lambda\alpha}) + (tc_{\alpha\beta} - b_{\alpha\beta})t^2 w_2 - \frac{1}{2}t^2(b^{\gamma}_{\alpha}\theta^{\epsilon}_{\gamma|\beta} + b^{\gamma}_{\beta}\theta^{\epsilon}_{\gamma|\alpha}),$$

we can see that the dominant term in the right hand side of this equation is $t\rho^{\epsilon}_{\alpha\beta}$. Therefore

$$\sum_{\alpha,\beta=1}^{2} \|\chi_{\alpha\beta}(\boldsymbol{v})\|^2 \gtrsim \epsilon^3 \, \|\underset{\approx}{\varrho}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)}.$$

We obtain

$$\|\boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^{\epsilon}} \gtrsim \epsilon^3 . \tag{6.5.18}$$

We then derive the upper bound on $\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|^2_{E^{\epsilon}} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^{\epsilon}}$. By the inequality (6.5.1), we have

$$\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|^2_{E^{\epsilon}} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^{\epsilon}} \lesssim \sum_{i,j=1}^{3} \|\varrho_{ij}\|^2 + \sum_{\alpha,\beta=1}^{2} \|\sigma_0^{\alpha\beta}\|^2 + \sum_{\alpha,\beta=1}^{2} \|\sigma_2^{\alpha\beta}\|^2. \tag{6.5.19}$$

From the equations (6.3.1), we have

$$\sigma_0^{\alpha\beta} = \frac{2}{3} H\,\epsilon^2\,\sigma_1^{\alpha\beta} + a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda}\,\epsilon^2\,P_o^3 a^{\alpha\beta},$$

$$\sigma_1^{\alpha\beta} = a^{\alpha\beta\lambda\gamma}\rho_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda}\,\epsilon^2 (P_e^3 + 2HP_o^3) a^{\alpha\beta},$$

$$\sigma_0^{3\alpha} = \frac{5}{4}[\mu a^{\alpha\beta}\tau_\beta^{\epsilon} - \epsilon^2\,P_o^{\alpha}],$$

and so, we have the estimates

$$\|\sigma_1^{\alpha\beta}\|^2 \lesssim \epsilon, \quad \|\sigma_0^{\alpha\beta}\|^2 \lesssim \epsilon^{3+2\theta}, \quad \|\sigma_0^{3\alpha}\|^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.20}$$

By the estimate (6.3.16), we have

$$\|\sigma_2^{\alpha\beta}\|^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.21}$$

From the equations (6.3.5) and (6.3.6), we have

$$\sigma_0^{33} = \frac{\epsilon^2}{2}(b_{\alpha\beta}\sigma_1^{\alpha\beta} + \epsilon^2\,P_e^{\alpha}|_\alpha - 2H\,\epsilon^2\,Q_a^3),$$

$$\sigma_1^{33} = \frac{\epsilon}{2}[b_{\alpha\beta}((\sigma_0^{\alpha\beta} - \frac{1}{3}\,\epsilon^2\,d_\gamma^\beta\sigma_1^{\alpha\gamma}) + \frac{2}{3} b_{\alpha\beta}(\sigma_2^{\alpha\beta} - \epsilon^2\,d_\gamma^\beta\sigma_1^{\alpha\gamma})$$
$$+ \epsilon^2\,P_o^{\alpha}|_\alpha + \epsilon^2\,P_e^3 + (1+\epsilon^2\,K)\,\epsilon^2\,Q_a^3],$$

and so the estimates

$$\|\sigma_0^{33}\|^2 \lesssim \epsilon^5, \quad \|\sigma_1^{33}\|^2 \lesssim \epsilon^{5+2\theta}. \tag{6.5.22}$$

Applying all the above estimates to the expression (6.3.28) of $\varrho_{\alpha\beta}$, it is readily seen that the square integral over ω^ϵ of every term is bounded by $O(\epsilon^5)$, except the term

$$(a_{\alpha\lambda} - tb_{\alpha\lambda})(a_{\beta\gamma} - tb_{\beta\gamma})\sigma_2^{\lambda\gamma} - \frac{\lambda}{2\mu + 3\lambda} g_{\alpha\beta} a_{\lambda\gamma} \sigma_2^{\lambda\gamma},$$

whose square integral on ω^ϵ, according to (6.5.21), is bounded by $O(\epsilon^{3+2\theta})$. Therefore we have

$$\|\varrho_{\alpha\beta}\|^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.23}$$

From the convergence (6.5.12), we know $\epsilon \|w_2\|_{H^1(\omega)} = \epsilon^\theta$, so $\|t^2 \partial_\alpha w_2\|^2 \lesssim \epsilon^{3+2\theta}$. Together with (6.5.20), we get

$$\|\varrho_{3\alpha}\|^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.24}$$

Our final concern is about ϱ_{33}. In the expression (6.3.30), the first term is

$$\frac{1}{E}(p_o^3 - \nu a_{\alpha\beta}\sigma_0^{\alpha\beta}) - w_1 = \frac{1}{E}(\epsilon^2 P_o^3 - \nu a_{\alpha\beta}\sigma_0^{\alpha\beta})$$

whose square integral over ω^ϵ is bounded, according to (6.5.20), by $O(\epsilon^{3+2\theta})$. The second term is, see (6.3.32),

$$\begin{aligned} t[\frac{1}{E}(p_e^3 - \nu a_{\alpha\beta}\sigma_1^{\alpha\beta}) - 2w_2] &= t[-2w_2 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\rho_{\alpha\beta}^0] - t\frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}(\rho_{\alpha\beta}^\epsilon - \rho_{\alpha\beta}^0) \\ &\quad + t\,\epsilon^2[\frac{1}{2\mu+\lambda} P_e^3 - \frac{2H\lambda^2}{\mu(2\mu+\lambda)(2\mu+3\lambda)} P_o^3]. \end{aligned}$$

By the convergence (6.5.12) and the estimate (6.5.16), we easily see that the square integral of this term on ω^ϵ is bounded by $O(\epsilon^{3+2\theta})$. The last term in (6.3.30) is also bounded, by using (6.5.21) and (6.5.22), by $O(\epsilon^{3+2\theta})$. We get

$$\|\varrho_{33}\|^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.25}$$

Therefore, by (6.5.20), (6.5.21), and (6.5.19), we have the upper bound

$$\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon}^2 + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}^2 \lesssim \epsilon^{3+2\theta}. \tag{6.5.26}$$

The conclusion of the theorem follows from the lower bound (6.5.18) and the upper bound (6.5.26) □

By replacing ϵ^θ and $\epsilon^{2\theta}$ with $o(1)$ in this proof, we will obtain a proof of Theorem 6.5.1.

Remark 6.5.2. *The estimate $\|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^{1+\theta}$ in (6.5.16) together with the convergence theorem furnishes a justification of the Kirchhoff–Love hypothesis in the case of flexural shells.*

6.5.3 Plate bending

If the shell is flat, the model (6.2.4) degenerates to the Reissner–Mindlin plate bending and stretching models analyzed in [2]. The limiting problem (6.5.3) combines the mixed formulation of the Kirchhoff–Love biharmonic plate bending model and the limiting plate stretching model. Under the loading assumption of this section, the solution of the limiting stretching equation is $\underset{\sim}{u}^0 = 0$. If the plate is totally clamped, the solution of the limiting problem is given by, see [8],

$$\underset{\sim}{\theta}^0 = -\underset{\sim}{\nabla} w^0, \quad \xi^0 = [0, \frac{E}{12(1-\nu^2)} \underset{\sim}{\nabla} \Delta w^0],$$

with w^0 given as the solution of the biharmonic equation. If the plate boundary is smooth, or is a convex polygon, and the loading function is smooth enough, such that the regularity $w^0 \in H^3$ holds, then we have $\xi^0 \in \underset{\approx}{L}_2^{\text{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)$, which is equivalent to V^*. Therefore, the index θ determined from (6.5.5) is 1.

It is readily seen that $a^{\alpha\beta}\rho^0_{\alpha\beta} = -\Delta w^0 \in H^1(\omega)$. By the standard cut-off argument, it can be shown that the index value θ determined from (6.5.11) is at least 1/2. Taking the minimum of these two values, the index value in (6.5.14) is at least 1/2, which gives the convergence rate of the Reissner–Mindlin plate bending model. This rate has already been shown to be optimal, see [16]. Therefore, Theorem 6.5.2 gives the best possible estimate for flexural shells.

Plate stretching and shear dominated plate bending are also special shell problems which are second kind membrane–shear shells, and will be remarked in the last section of this chapter.

6.6 Totally clamped elliptic shells

For totally clamped elliptic shells, convergence of the model solution toward the 3D solution in the relative energy norm can be proved under the loading assumption

(6.2.9). Convergence rate will be determined and attributed to the regularity of the solution of the ϵ-independent limiting membrane shell model. This regularity is defined in terms of interpolation spaces. The rate $O(\epsilon^{1/6})$ will be established if the shell data are smooth enough in the usual Sobolev sense.

A shell Ω^ϵ is elliptic, if its middle surface S is uniformly elliptic in the sense that the Gauss curvature K is strictly positive. I.e., there exists a $K_0 > 0$, such that $K \geq K_0$. For any given $t \in (-\epsilon, \epsilon)$, We define $S(t) = \{\mathbf{\Phi}(\underset{\sim}{x}, t) | \underset{\sim}{x} \in \omega)\}$, which is a surface parallel to the middle surface S and at the height t. Let $K(t)$ be the Gauss curvature of $S(t)$, from (4.1.6). It is easy to see that $K(t) = K/(1 - 2tH + t^2 K)$. So if S is elliptic, $S(t)$ is elliptic if t is small enough.

We assume the shell is totally clamped, So $\partial_D \omega = \partial \omega$ and $\partial_D \omega^\epsilon = \partial \omega \times [-\epsilon, \epsilon]$. The space H then is $\underset{\sim}{H}{}^1_0(\omega) \times \underset{\sim}{H}{}^1_0(\omega) \times H^1_0(\omega)$. Under some smoothness assumption on the shell middle surface S, the following Korn-type inequality was established in [23] and [19]: There exists a constant C such that for any $\underset{\sim}{u} \in \underset{\sim}{H}{}^1_0(\omega)$, $w \in L_2(\omega)$

$$\|\underset{\sim}{u}\|^2_{\underset{\sim}{H}^1(\omega)} + \|w\|^2_{L_2(\omega)} \leq C \|\underset{\approx}{\gamma}(\underset{\sim}{u}, w)\|^2_{\underset{\approx}{L}{}_2^{\text{sym}}(\omega)}, \tag{6.6.1}$$

where $\underset{\approx}{\gamma}(\underset{\sim}{u}, w)$ is the membrane strain engendered by the displacement $u_\alpha \boldsymbol{a}^\alpha + w \boldsymbol{a}^3$ on the middle surface, see (6.2.3). It was shown in [58] that this inequality is valid only on totally clamped elliptic shells. Applying this inequality to the surface $S(t)$, we get

$$\|\underset{\sim}{u}\|^2_{\underset{\sim}{H}^1(\omega)} + \|w\|^2_{L_2(\omega)} \leq C(t) \|\chi_{\alpha\beta}(\underset{\sim}{u}, w)\|^2_{\underset{\approx}{L}{}_2^{\text{sym}}(\omega)}.$$

$\chi_{\alpha\beta}(\underset{\sim}{u}, w)$ is the tangential part of the 3D strain χ_{ij} engendered by a displacement whose restriction on $S(t)$ is $u_\alpha \boldsymbol{g}^\alpha + w\boldsymbol{g}^3$. If ϵ is small enough, this inequality uniformly holds for all $t \in [-\epsilon, \epsilon]$.

Taking integration at both sides of the above inequality with respect to t, we see that there exists a constant $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon^0$, for any displacement field $\boldsymbol{v} = v_i \in \boldsymbol{H}^1_D(\omega^\epsilon)$, we have

$$\sum_{\alpha=1}^{2} \|v_\alpha\|^2_{H^1(\omega)\times L_2(-\epsilon,\epsilon)} + \|v_3\|^2_{L_2(\omega^\epsilon)} \lesssim \sum_{i,j=1}^{3} \|\chi_{ij}(\boldsymbol{v})\|^2_{L_2(\omega^\epsilon)}. \tag{6.6.2}$$

Comparing this inequality to that given in Theorem 6.3.5, which is valid for all shells, we see that the particularity of totally clamped elliptic shells is remarkable.

As what we did for the flexural shells, we first resolve the term r in the identity (6.3.17). Using the expression (6.3.18), we have

$$|r| \lesssim \epsilon[\|\underset{\approx}{\sigma}0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)} + \|\underset{\approx}{\sigma}2\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}][\|\underset{\sim}{v}^* - \underset{\sim}{v}\|_{\underset{\sim}{H}^1(\omega)\times \underset{\sim}{L}_2(-\epsilon,\epsilon)} + \|v_3^* - v_3\|_{L_2(\omega^\epsilon)}].$$

The inequality (6.6.2) allows us to give the factor ϵ to the first half of the above right hand side. Using the inequality (6.6.2) and Cauchy's inequality to the identity (6.3.17), we get

$$\begin{aligned}
&\int_{\Omega^\epsilon} A_{ijkl}(\sigma^{kl} - \sigma^{*kl})(\sigma^{ij} - \sigma^{*ij}) + \int_{\Omega^\epsilon} C^{ijkl}[\chi_{kl}(\boldsymbol{v}) - \chi_{kl}(\boldsymbol{v}^*)][\chi_{ij}(\boldsymbol{v}) - \chi_{ij}(\boldsymbol{v}^*)] \\
&\lesssim \int_{\Omega^\epsilon} [A_{ijkl}\sigma^{kl} - \chi_{ij}(\boldsymbol{v})][\sigma^{ij} - C^{ijkl}\chi_{kl}(\boldsymbol{v})] + \epsilon^2 \|\underset{\approx}{\sigma}0\|^2_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)} + \epsilon^2 \|\underset{\approx}{\sigma}2\|^2_{\underset{\approx}{L}_2^{\text{sym}}(\omega^\epsilon)}.
\end{aligned} \tag{6.6.3}$$

6.6.1 Reformulation of the resultant loading functional

From the inequality (6.6.1), it is immediately seen that for totally clamped elliptic shells, we have $\ker B = 0$. Therefore, no matter what is the resultant loading functional in the model, the shell problem can never be flexural. Theorem 3.3.4, Theorem 3.3.5, (3.4.6), and (3.4.7) in Chapter 3 are the right tools to analyze the asymptotic behavior of the model solution. According to the classification of Section 3.5, if the condition (6.6.4) below is satisfied, the totally clamped elliptic shell problem is of membrane–shear.

Since $\ker B = 0$ and B is surjective from H to W, by the closed range theorem, there exists a $\zeta_*^0 \in W^*$, such that the leading term of the resultant loading functional can be equivalently written as

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \langle \zeta_*^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H.$$

We recall that without further assumption, the solution of the model problem is untractable. The condition we imposed in Chapter 3 is $\zeta_*^0 \in V^*$. Under this condition, the loading functional can be further written as

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \langle \zeta_*^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = (\zeta^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z))_V, \tag{6.6.4}$$

here, $\zeta^0 \in V$ is the Riesz representation of $\zeta_*^0 \in V^*$. Therefore the condition (6.6.4) is equivalent to the existence of $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V = V_0 \times \underset{\sim}{L}_2(\omega)$, such that

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \int_\omega a^{\alpha\beta\lambda\gamma}\gamma^0_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu\int_\omega a^{\alpha\beta}\tau^0_\beta\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}. \tag{6.6.5}$$

Recall that the expression of the leading term in the loading functional is

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{5}{6}\int_\omega p_o^\alpha \tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} - \frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$+ \int_\omega [(p_e^\alpha + q_a^\alpha - 2b_\gamma^\alpha p_o^\gamma)y_\alpha + (p_o^\alpha|_\alpha + p_e^3 + q_a^3)z]\sqrt{a}d\underset{\sim}{x}. \quad (6.6.6)$$

Comparing this expression to (6.6.5), we just need to choose

$$\tau_\alpha^0 = \frac{1}{\mu}a_{\alpha\beta}p_o^\beta, \quad (6.6.7)$$

obviously, $\underset{\sim}{\tau}^0 \in \underset{\sim}{L}_2(\omega)$.

Thanks to the inequality (6.6.1), we know that $\underset{\approx}{\gamma}$ defines an isomorphism between the space $\underset{\sim}{H}_0^1(\omega)\times L_2(\omega)$ and a closed subspace of $\underset{\approx}{L}_2^{\text{sym}}(\omega)$, which should be V_0. Since the last two terms in (6.6.6) together define a continuous linear functional on $\underset{\sim}{H}_0^1(\omega)\times L_2(\omega)$, by the Riesz representation theorem, there exists a unique $(\underset{\sim}{u}^0, w^0) \in \underset{\sim}{H}_0^1(\omega)\times L_2(\omega)$ such that

$$\underset{\approx}{\gamma}^0 = \underset{\approx}{\gamma}(\underset{\sim}{u}^0, w^0) \in V_0 \quad (6.6.8)$$

and

$$\int_\omega a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}(\underset{\sim}{u}^0, w^0)\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} = -\frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$+ \int_\omega [(p_e^\alpha + q_a^\alpha - 2b_\gamma^\alpha p_o^\gamma)y_\alpha + (p_o^\alpha|_\alpha + p_e^3 + q_a^3)z]\sqrt{a}d\underset{\sim}{x}$$
$$\forall\ (\underset{\sim}{y}, z) \in \underset{\sim}{H}_0^1(\omega)\times L_2(\omega). \quad (6.6.9)$$

Therefore, (6.6.7) and (6.6.8) together reformulated the resultant loading functional in the desired way (6.6.5).

Note that the equation (6.6.9) can be viewed as an equation to determine the functions $(\underset{\sim}{u}^0, w^0) \in \underset{\sim}{H}_0^1(\omega) \times L_2(\omega)$. This is formally the same as the limiting elliptic membrane shell model of [18], but note that the right hand side is different. Here, the odd part of the surface force p_o^i is incorporated.

6.6.2 Asymptotic behavior of the model solution

From Theorem 3.3.5 and (3.4.7), we get the asymptotic behavior of the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$:

$$\epsilon \,\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{6.6.10}$$

If we assume more regularity on $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0)$, say,

$$(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in [W, V]_{1-\theta,q} \tag{6.6.11}$$

for some $\theta \in (0, 1)$ and $q \in [1, \infty]$, or $\theta \in [0, 1]$ and $q \in (1, \infty)$, by Theorem 3.3.4 and (3.4.6), we get the stronger estimate of the asymptotic behavior of the model solution:

$$\epsilon \,\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\text{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) \lesssim \epsilon^\theta . \tag{6.6.12}$$

We assume that $\underset{\approx}{\gamma}^0$ and $\underset{\sim}{\tau}^0$ can not be zero simultaneously, otherwise $\boldsymbol{f}_0 = 0$. Under some further assumptions on the smoothness of loading functions, with a similar but more tedious analysis, we can prove the convergence of the model solution to the 3D solution even if $\boldsymbol{f}_0 = 0$.

Based on the requirements imposed on the correction functions w_1 and w_2, and recalling the expressions (6.3.31) and (6.3.32), which need to be small, we define w_1 as the solution of the equation

$$\begin{gathered}\epsilon^2(\nabla w_1, \nabla v)_{\underset{\sim}{L}_2(\omega)} + (w_1, v)_{L_2(\omega)} = (-\frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o, v)_{L_2(\omega)}, \\ w_1 \in H^1_0(\omega), \quad \forall\, v \in H^1_0(\omega)\end{gathered} \tag{6.6.13}$$

and define

$$w_2 = 0. \tag{6.6.14}$$

The explanation of this choice of the correction functions is right in contrast to what we made for flexural shells on page 181.

From the definition (6.6.8) of $\underset{\approx}{\gamma}^0$, we see that $\underset{\approx}{\gamma}^0 \in \underset{\approx}{L}_2^{\text{sym}}(\omega)$, so $a^{\alpha\beta}\gamma^0_{\alpha\beta} \in L_2(\omega)$. By (3.3.38) in Theorem 3.3.6, we have the convergence

$$\epsilon\,\|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o\|_{L_2(\omega)} \to 0 \ \ (\epsilon \to 0). \tag{6.6.15}$$

If we assume

$$\lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o \in [H^1_0(\omega), L_2(\omega)]_{1-\theta,p} \tag{6.6.16}$$

for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or $\theta \in [0,1]$ and $p \in (1,\infty)$, by (3.3.36) in Theorem 3.3.6, we have,

$$\epsilon \|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o \|_{L_2(\omega)}$$

$$\lesssim K(\epsilon, \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_0(\omega)]) \lesssim \epsilon^\theta \,. \quad (6.6.17)$$

The values of the index θ in (6.6.11) and (6.6.16) might be different. We choose the least one so that the convergences (6.6.12) and (6.6.17) hold simultaneously.

From the asymptotic estimates (6.6.10) we see

$$\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} \lesssim o(\epsilon^{-1}), \quad \|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} \lesssim 1, \quad \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim 1.$$

Under the regularity assumption (6.6.11), we have

$$\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} \lesssim \epsilon^{\theta-1}, \quad \|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} \lesssim 1, \quad \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim 1.$$

By the equivalency (6.2.8) and the inequality (6.6.1), we get the *a priori* estimates

$$\|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} \lesssim 1, \quad \|w^\epsilon\|_{L_2(\omega)} \lesssim 1, \quad \|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} \lesssim o(\epsilon^{-1}) \text{ or } O(\epsilon^{\theta-1}), \text{ if (6.6.11)},$$

$$\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^{-1}(\omega)} \lesssim 1, \quad \|w^\epsilon\|_{H^1(\omega)} \lesssim \|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} + \|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{L}_2(\omega)}. \quad (6.6.18)$$

6.6.3 Convergence theorems

The convergence of the 2D model solution to the 3D solution can be proved if the loading functions satisfy the condition (6.2.9). Under further assumption on the regularity of $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0)$, convergence rate can be established.

THEOREM 6.6.1. *Let $\boldsymbol{v}^*$ and $\boldsymbol{\sigma}^*$ be the 3D solution of the shell problem, $\boldsymbol{v}$ the displacement defined by the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ together with the correction functions w_1, w_2 defined in (6.6.13) and (6.6.14) through the formulae (6.3.13), and $\boldsymbol{\sigma}$ the stress field defined by (6.3.7). Under the condition (6.2.9), we have the convergence*

$$\lim_{\epsilon \to 0} \frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0. \tag{6.6.19}$$

If $\underset{\approx}{\gamma}^0 = \underset{\approx}{\gamma}(\underset{\sim}{u}^0, w^0)$ and $\tau_\alpha^0 = \dfrac{1}{\mu} a_{\alpha\beta} p_o^\beta$ satisfy the regularity condition

$$(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in [W, V]_{1-\theta, q} \quad \text{and} \quad \lambda a^{\alpha\beta} \gamma_{\alpha\beta}^0 - p_o^3 \in [H_0^1(\omega), L_2(\omega)]_{1-\theta, p} \tag{6.6.20}$$

for some $\theta \in (0,1)$ and $p, q \in [1, \infty]$, or $\theta \in [0,1]$ and $p, q \in (1, \infty)$, then the convergences (6.6.12) and (6.6.17) hold simultaneously, and we have

THEOREM 6.6.2. *If the regularity condition (6.6.20) is satisfied, we have the convergence rate*

$$\frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^\theta. \tag{6.6.21}$$

With all the preparations of the last subsection, the proofs of these theorems are almost the same as that of Theorem 5.5.1 for spherical shells, except that now we need to use the inequality (6.6.3) rather than the two energies principle. For this reason, and for lack of space, the proofs are omited.

The regularity condition (6.6.20) is not easy to interpret. We just give an unrealistic example to explain its meaning. We will determine the index θ in the next subsection under some smoothness assumption on the shell data in the usual sense.

Let the shell be loaded in such a special way that in the reformulation of the loading functional (6.6.5), $\underset{\approx}{\gamma}^0$ and $\underset{\sim}{\tau}^0$ are given by

$$\gamma^0_{\alpha\beta} = \gamma_{\alpha\beta}(\underset{\sim}{u}^\circ, w^\circ) \quad \text{and} \quad \frac{1}{\mu} a_{\alpha\beta} p^\beta_o = \tau^0_\alpha = \tau_\alpha(\underset{\sim}{\theta}^\circ, \underset{\sim}{u}^\circ, w^\circ),$$

with $\underset{\sim}{\theta}^\circ \in \underset{\sim}{H}^1_0(\omega), w^\circ \in H^1_0(\omega)$, and $\underset{\sim}{u}^\circ \in \underset{\sim}{H}^1_0(\omega) \cap \underset{\sim}{H}^2(\omega)$. We assume $p^3_o \in H^1(\omega)$. It is easy to see that $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in W$, so the index θ determined from (6.6.11) is equal to 1. Since $\lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o \in H^1(\omega)$, by the standard cut-off argument, the index θ determined from (6.6.16) is at least 1/2. The convergence rate then is determined by the smaller one of these two values. I.e., at least $\epsilon^{1/2}$.

6.6.4 Estimates of the K-functional for smooth data

We have seen in the last subsection that the convergence rate of the model solution to the 3D solution in the relative energy norm is determined by the the values of the K-functionals in (6.6.12) and (6.6.17). In this subsection we estimate these values for the elliptic shell under the assumption that the shell boundary, middle surface, and loading functions are smooth enough.

Based on the definitions of the spaces W, V, and the K-functional (6.5.7), the two K-functionals involved in the convergence can be equivalently expressed as

$$K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) = \inf_{(\underset{\approx}{\gamma}, \underset{\sim}{\tau}) \in W} [\|(\underset{\approx}{\gamma}^0 - \underset{\approx}{\gamma}, \underset{\sim}{\tau}^0 - \underset{\sim}{\tau})\|_V + \epsilon \, \|(\underset{\approx}{\gamma}, \underset{\sim}{\tau})\|_W]$$

$$= \inf_{(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H} [\|(\underset{\approx}{\gamma}^0 - \underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}^0 - \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w))\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega) \times \underset{\sim}{L}_2(\omega)} + \epsilon \, \|(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_H] \tag{6.6.22}$$

and

$$K(\epsilon, \lambda a^{\alpha\beta} \gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_0(\omega)]) = \inf_{w \in H^1_0(\omega)} [\|\lambda a^{\alpha\beta} \gamma^0_{\alpha\beta} - p^3_o - w\|_{L_2(\omega)} + \epsilon \, \|w\|_{H^1_0(\omega)}]. \tag{6.6.23}$$

The strategy to determine the K-functional values is to make a good choice for $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in \underset{\sim}{H}^1_0(\omega) \times \underset{\sim}{H}^1_0(\omega) \times H^1_0(\omega)$ in the former and a good choice for $w \in H^1_0(\omega)$ in the latter so that the infimums can be roughly reached. This can be done by doing a little more delicate cut-off argument, which requires some regularity results.

We assume the following smoothness on the shell data: The shell boundary $\gamma = \partial S \in C^4$. The loading functions $p^\alpha_o \in H^3(\omega)$, $p^\alpha_e \in H^1(\omega)$, $p^3_o \in H^2(\omega)$, $p^3_e \in H^2(\omega)$, $q^\alpha_a \in H^1(\omega)$, $q^3_a \in H^2(\omega)$.

Lemma 6.6.3. *Under this assumption, the solution of the equation (6.6.9) has the regularity*

$$\underset{\sim}{u}^0 \in H^3(\omega) \cap H^1_0(\omega) \quad \textit{and} \quad w^0 \in H^2(\omega).$$

This lemma follows from a more general regularity theorem on the solution of the limiting membrane shell model in [27]. Under the above smoothness assumption on the data, we have the regularity

$$\gamma^0_{\alpha\beta} = \gamma_{\alpha\beta}(\underset{\sim}{u}^0, w^0) \in H^2(\omega), \quad \tau^0_\alpha = \frac{1}{\mu} a_{\alpha\beta} p^\beta_o \in H^3(\omega), \quad \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o \in H^2(\omega). \tag{6.6.24}$$

We also need the following cut-off lemma.

LEMMA 6.6.4. *Let $\omega \subset \mathbb{R}^2$ be an open connected domain, and $\partial\omega \in C^2$. Let $\alpha > 0$ and $\epsilon > 0$ be two positive numbers. Then for any $f \in H^1(\omega)$, there exists a $f^0 \in H^1_0(\omega)$ such that*

$$\|f - f^0\|_{L_2(\omega)} \le \epsilon^\alpha \|f\|_{L_2(\omega)}, \quad \|f^0\|_{H^1(\omega)} \le \epsilon^{-\alpha} \|f\|_{H^1(\omega)}.$$

If $f \in H^2(\omega)$, we further have $\|f^0\|_{H^2(\omega)} \le \epsilon^{-3\alpha} \|f\|_{H^2(\omega)}$.

The proof of this lemma can be found in [36]. An equivalent result can be found in [43]. With these preparations, we can prove

THEOREM 6.6.5. *Under the above smoothness assumption on the shell data, we have the following estimates on the K-functionals:*

$$K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) \lesssim \epsilon^{1/6} \tag{6.6.25}$$

and

$$K(\epsilon, \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_0(\omega)]) \lesssim \epsilon^{1/2}\,. \tag{6.6.26}$$

Therefore the value of the index θ is at least $1/6$, which gives, by Theorem 6.6.2, the convergence rate of the shell model.

Proof. According to (6.6.22), we need to estimate

$$\inf_{(\underset{\sim}{\theta},\underset{\sim}{u},w)\in H}[\|(\underset{\approx}{\gamma}(\underset{\sim}{u}^0,w^0)-\underset{\approx}{\gamma}(\underset{\sim}{u},w),\underset{\sim}{\tau}^0-\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w))\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)\times\underset{\sim}{L}_2(\omega)}+\epsilon\,\|(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_H].$$

Since $\underset{\sim}{u}^0\in\underset{\sim}{H}_0^1(\omega)$, we can choose $\underset{\sim}{u}=\underset{\sim}{u}^0$, so we have

$$\gamma_{\alpha\beta}(\underset{\sim}{u}^0,w^0)-\gamma_{\alpha\beta}(\underset{\sim}{u},w)=b_{\alpha\beta}(w-w^0).$$

Taking a positive number a, since $w^0\in H^2(\omega)$, by Lemma 6.6.4, there exists a $w\in H_0^1(\omega)\cap H^2(\omega)$ such that

$$\|w^0-w\|_{L_2(\omega)}\le\epsilon^a\,\|w^0\|_{L_2(\omega)},\quad \|w\|_{H^1(\omega)}\le\epsilon^{-a}\,\|w^0\|_{H^1(\omega)},$$

$$\|w\|_{H^2(\omega)}\le\epsilon^{-3a}\,\|w^0\|_{H^2(\omega)}.$$

From the definition (6.2.3), we have $\tau_\alpha(\underset{\sim}{\theta},\underset{\sim}{u},w)=\theta_\alpha+\partial_\alpha w+b_\alpha^\lambda u_\lambda$. Let b be a positive numberi. By Lemma 6.6.4, for the above chosen $\underset{\sim}{u}$ and w, there exists a $\underset{\sim}{\theta}\in\underset{\sim}{H}_0^1(\omega)$ such that

$$\begin{aligned}\|\underset{\sim}{\tau}^0-\underset{\sim}{\tau}(\underset{\sim}{\theta},\underset{\sim}{u},w)\|_{\underset{\sim}{L}_2(\omega)}&\simeq\sum_{\alpha=1}^2\|\theta_\alpha+\partial_\alpha w+b_\alpha^\lambda u_\lambda-\tau_\alpha^0\|_{L_2(\omega)}\\&\le\epsilon^b\sum_{\alpha=1}^2\|\partial_\alpha w+b_\alpha^\lambda u_\lambda-\tau_\alpha^0\|_{L_2(\omega)}\\&\le\epsilon^{b-a}\,\|w^0\|_{H^1(\omega)}+\epsilon^b\,\|\underset{\sim}{u}^0\|_{\underset{\sim}{L}_2(\omega)}+\epsilon^b\,\|\underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)}\end{aligned}$$

and

$$\|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1(\omega)} \le \epsilon^{-b} \sum_{\alpha=1}^{2} \|\partial_\alpha w + b_\alpha^\lambda u_\lambda - \tau_\alpha^0\|_{H^1(\omega)}$$
$$\le \epsilon^{-b-3a} \|w^0\|_{H^2(\omega)} + \epsilon^{-b} \|\underset{\sim}{u}^0\|_{\underset{\sim}{H}^1(\omega)} + \epsilon^{-b} \|\underset{\sim}{\tau}^0\|_{\underset{\sim}{H}^1(\omega)}.$$

With these $(\underset{\sim}{\theta}, \underset{\sim}{u}, w)$ substituted in the arguments of the infimum, we see

$$K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) \lesssim \epsilon^a \|w^0\|_{L_2(\omega)} + \epsilon^{b-a} \|w^0\|_{H^1(\omega)} + \epsilon^b \|\underset{\sim}{u}^0\|_{\underset{\sim}{L}_2(\omega)}$$
$$+ \epsilon^b \|\underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} + \epsilon^{1-b-3a} \|w^0\|_{H^2(\omega)} + (\epsilon^{1-b} + \epsilon)\|\underset{\sim}{u}^0\|_{\underset{\sim}{H}^1(\omega)}$$
$$+ \epsilon^{1-b} \|\underset{\sim}{\tau}^0\|_{\underset{\sim}{H}^1(\omega)} + \epsilon^{1-a} \|w^0\|_{H^1(\omega)}.$$

Note that $\underset{\sim}{\tau}^0$, $\underset{\sim}{u}^0$ and w^0 are all ϵ-independent functions. The best values for a and b should make $a = b - a = 1 - b - 3a$, and are given by $a = 1/6, b = 1/3$. We obtain

$$K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V, W]) \lesssim \epsilon^{1/6}. \tag{6.6.27}$$

The proof of (6.6.26) is simpler and so ignored. □

Based on this estimate, we can compare the strain energy that can be sustained by a totally clamped spherical shell with that which can be sustained by a totally clamped flexural plate. The former is a special totally clamped elliptic shell, and the latter is a special flexural shell. For the plate, by (6.5.18), the strain energy is $O(\epsilon^3)$, and the model solution tends to a finite limit in the space H. For spherical shell, by (5.5.24), the strain energy is $O(\epsilon)$, but by the estimate (6.6.18), the H norm of the solution is only bounded by $O(\epsilon^{-5/6})$. To keep the solution bounded, we have to reduce the loads by multiplying a factor of $O(\epsilon^{5/6})$. The strain energy will be scaled to $O(\epsilon[\epsilon^{5/6}]^2) = O(\epsilon^{8/3})$. Therefore, without blowing up in displacement, the strain energy that can be sustained by a totally clamped spherical shell is $O(\epsilon^{-1/3})$ times that can be sustained by a plate.

If $\tau_\alpha^0 = \dfrac{1}{\mu} a_{\alpha\beta} p_o^\beta = 0$, we can make another choice of $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$ in the proof of Theorem 6.6.5 and prove

$$K(\epsilon, (\underset{\approx}{\gamma}^0, 0), [V, W]) \lesssim \epsilon^{1/5}. \tag{6.6.28}$$

This can be done by letting $\underset{\sim}{u} = \underset{\sim}{u}^0$, choosing $w \in H_0^2(\omega)$ such that

$$\|w - w^0\|_{L_2(\omega)} \lesssim \epsilon^{1/5}, \quad \|w\|_{H^1(\omega)} \lesssim \epsilon^{-1/5}, \quad \|w\|_{H^2(\omega)} \lesssim \epsilon^{-4/5},$$

and taking $\theta_\alpha = -\partial_\alpha w - b_\alpha^\lambda u_\lambda$. The existence of such w can be proved by using a lemma of [45]. Therefore, if the odd part of the tangential surface forces vanishes, the model convergence rate is $O(\epsilon^{1/5})$ in the relative energy norm.

The estimate (6.6.12) not only plays a crucial role in establishing the convergence rate of the model, but also gives an estimate on the difference between our model solution and the solution $(\underset{\sim}{u}^0, w^0)$ of the limiting model (6.6.9). The K-functional value will be used to prove the convergence rate of the limiting model solution in Section 7.4.

As we have mentioned at the end of the last subsection, the convergence rate can be as high as $\epsilon^{1/2}$, but we can only prove this when the loading functions are special.

If we only assume the smoothness on the shell data in the usual Sobolev sense, under the most general loading assumption, the convergence rate $O(\epsilon^{1/6})$ is the best we can prove. It seems possible to get better results by other methods, see [33], [52], [53], [26], and [39].

REMARK 6.6.1. *The convergence* $\|\underset{\sim}{\tau}^{\epsilon} - \underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}_2(\omega)} \to 0$ $(\epsilon \to 0)$ *in (6.6.10), or the estimate* $\|\underset{\sim}{\tau}^{\epsilon} - \underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^{\theta}$ *in (6.6.12), together with the expression (6.6.7) and Theorems 6.6.1 and 6.6.2 violate the Kirchhoff–Love hypothesis if the odd part of the tangential surface force* p_o^{α} *is not zero. This is in sharp contrast to the case of flexural shells for which the Kirchhoff–Love assumption can always be proved, see Remark 6.5.2. On the other hand, these convergences furnish a proof for this hypothesis if, say, the odd part of the tangential surface force vanished, cf., [40].*

6.7 Membrane–shear shells

The totally clamped elliptic shells we discussed in the previous section are special examples of general membrane–shear shells defined in Section 3.5 and Section 6.4. There are two major difficulties in general situation. One lies in the reformulated resultant loading functional:

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \langle \zeta_*^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H, \tag{6.7.1}$$

with $\zeta_*^0 \in W^*$. To apply the abstract theory of Chapter 3 to analyze the asymptotic behavior of the model solution, we need to assume $\zeta_*^0 \in V^*$. This condition, which was unconditionally satisfied by totally clamped elliptic shells, now imposes a stringent restriction on the resultant loading functional.

Another difficulty, which is even more formidable, lies in resolving the extra term r in the integration identity (6.3.17). This identity, as in the last two sections, plays the keystone role in the model justification. We can neither resolve this extra term in the way of handling totally clamped elliptic shells, see (6.6.3), since the ϵ-independent Korn-type inequality (6.6.2) is no longer valid, nor can we resort to the measure for flexural shells, see (6.5.1), because the quantity $\|\underset{\approx}{\sigma}0\|^2_{L_2^{\mathrm{sym}}(\omega^\epsilon)}$ is not small any more.

Both of these difficulties will be eluded by imposing further conditions. The formulations and proofs of convergence theorems will otherwise be the same as those in the last section. The shell problems that are ruled out by these conditions abound, for which the convergence of the model solutions to the 3D solutions might not hold in the relative energy norm.

6.7.1 Asymptotic behavior of the model solution

In this subsection, we interpret the abstractly imposed condition $\zeta^0_* \in V^*$ for general membrane–shear shell problems, and analyze the asymptotic behavior of the model solution by using the abstract theory of Chapter 3.

The condition $\zeta^0_* \in V^*$ is equivalent to the existence of $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V$, such that $\zeta^0 = (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0)$, and

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = (\zeta^0, B(\underset{\sim}{\phi}, \underset{\sim}{y}, z))_V$$

$$= \int_\omega a^{\alpha\beta\lambda\gamma}\gamma^0_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu\int_\omega a^{\alpha\beta}\tau^0_\beta\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H. \tag{6.7.2}$$

Recalling the expression (6.2.5)

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \frac{5}{6}\int_\omega p^\alpha_o\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} - \frac{\lambda}{2\mu+\lambda}\int_\omega p^3_o a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$

$$+ \int_\omega [(p^\alpha_e + q^\alpha_a - 2b^\alpha_\gamma p^\gamma_o)y_\alpha + (p^\alpha_o|_\alpha + p^3_e + q^3_a)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p^\alpha_0 y_\alpha \tag{6.7.3}$$

and the factorization (6.4.1) of the space $V = V_0 \times L_2(\omega)$, we see that the requirement (6.7.2) is equivalent to the following two requirements. First,

$$p^\alpha_o \in L_2(\omega) \quad \text{and} \quad p^3_o \in L_2(\omega). \tag{6.7.4}$$

Second, there exists a $\underset{\approx}{\kappa} \in V_0$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\kappa_{\lambda\gamma}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} = \int_\omega [(p^\alpha_e + q^\alpha_a - 2b^\alpha_\gamma p^\gamma_o)y_\alpha + (p^\alpha_o|_\alpha + p^3_e + q^3_a)z]\sqrt{a}d\underset{\sim}{x}$$

$$+ \int_{\gamma_T} p^\alpha_0 y_\alpha \forall\ (\underset{\sim}{y}, z) \in \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega). \tag{6.7.5}$$

Note that the second term in the right hand side of (6.7.3) can be equally written as

$$-\frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x} = -\frac{\lambda}{2\mu(2\mu+3\lambda)}\int_\omega a^{\alpha\beta\lambda\gamma}a_{\lambda\gamma}p_o^3\gamma_{\alpha\beta}(\underset{\sim}{y},z)\sqrt{a}d\underset{\sim}{x}. \tag{6.7.6}$$

Therefore, if $p_o^3 \in L_2(\omega)$, we can determine $\underset{\approx}{\gamma}^0 \in V_0$ as

$$\gamma^0_{\alpha\beta} = \kappa_{\alpha\beta} - \frac{\lambda}{2\mu(2\mu+3\lambda)}P_{V_0}(a_{\alpha\beta}p_o^3), \tag{6.7.7}$$

where P_{V_0} is the orthogonal projection from $\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)$ to V_0. By defining

$$\tau^0_\alpha = \frac{1}{\mu}a_{\alpha\beta}p_o^\beta, \tag{6.7.8}$$

we obtain $\zeta^0 = (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in V$ such that the loading functional be reformulated as (6.7.2).

Under the condition (6.7.2), the asymptotic behavior of the model solution then follows from Theorem 3.3.5 and (3.4.7). We have

$$\epsilon\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{6.7.9}$$

If we assume more regularity on $(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0)$, say,

$$(\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0) \in [W, V]_{1-\theta,q} \tag{6.7.10}$$

for some $\theta \in (0,1)$ and $q \in [1,\infty]$, or $\theta \in [0,1]$ and $q \in (1,\infty)$, by Theorem 3.3.5 and (3.4.6), we have the stronger estimate of the asymptotic behavior of the model solution:

$$\epsilon \|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim K(\epsilon, (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0), [V,W]) \lesssim \epsilon^\theta . \tag{6.7.11}$$

From the asymptotic estimate (6.7.9), we see

$$\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim o(\epsilon^{-1}), \quad \|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim 1, \quad \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim 1.$$

Under the regularity assumption (6.7.10), by (6.7.11), we have

$$\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^{\theta-1}, \quad \|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim 1, \quad \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim 1.$$

By the equivalency (6.2.8), we get the *a priori* estimates

$$\|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} + \|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} + \|w^\epsilon\|_{H^1(\omega)} \lesssim o(\epsilon^{-1})$$

$$\text{(or } \|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} + \|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{H}^1(\omega)} + \|w^\epsilon\|_{H^1(\omega)} \lesssim \epsilon^{\theta-1}, \quad \text{if the regularity (6.7.10) holds),}$$

$$\|w^\epsilon\|_{H^1(\omega)} \lesssim \|\underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} + \|\underset{\sim}{u}^\epsilon\|_{\underset{\sim}{L}_2(\omega)}. \tag{6.7.12}$$

These estimates are much weaker than those for totally clamped elliptic shells, see (6.6.18), because of lack of the Korn-type inequality (6.6.1), which is a characterization of totally clamped elliptic shells.

The two conditions (6.7.4) and (6.7.5) together are equivalent to the condition (6.7.2). The first condition (6.7.4) is trivially satisfied, while the second one (6.7.5), i.e., the existence of $\underset{\approx}{\kappa} \in V_0$ such that (6.7.5) holds, can be connected to the "generalized membrane shell" theory, see [20] and [18], in the following way.

The membrane strain operator $\underset{\approx}{\gamma}(\underset{\sim}{y}, z)$ defines a linear continuous operator

$$\underset{\approx}{\gamma} : \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega) \longrightarrow V_0,$$

whose range is dense in V_0. We first consider the case of $\ker \underset{\approx}{\gamma} = 0$. In this case, $\|\underset{\approx}{\gamma}(\underset{\sim}{y}, z)\|_{V_0}$ defines a norm on the space $\underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$, which is weaker than the original norm. In the notation of [18], we denote the completion of $\underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$ with respect to this new norm by $V^\sharp_M(\omega)$. Obviously, $\underset{\approx}{\gamma}$ can be uniquely extended to $V^\sharp_M(\omega)$, and the extended linear continuous operator, still denoted by $\underset{\approx}{\gamma}$, defines an isomorphism between $V^\sharp_M(\omega)$ and V_0. By the closed range theorem, for any $f \in [V^\sharp_M(\omega)]^*$, there exists a unique $\underset{\approx}{\kappa} \in V_0$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma} \kappa_{\lambda\gamma} \gamma_{\alpha\beta}(\underset{\sim}{y}, z) \sqrt{a}\, d\underset{\sim}{x} = \langle f, (\underset{\sim}{y}, z) \rangle \quad \forall\ (\underset{\sim}{y}, z) \in V^\sharp_M(\omega). \tag{6.7.13}$$

Therefore, the problem of existence of $\underset{\approx}{\kappa} \in V_0$ in (6.7.5) is equivalent to the problem that whether or not the linear functional

$$\int_\omega [(p^\alpha_e + q^\alpha_a - 2 b^\alpha_\gamma p^\gamma_o) y_\alpha + (p^\alpha_o|_\alpha + p^3_e + q^3_a) z] \sqrt{a}\, d\underset{\sim}{x} + \int_{\gamma_T} p^\alpha_0 y_\alpha,$$

which is defined on the space $\underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$ by the right hand side of (6.7.5), can be extended to a linear continuous functional on the space $V^\sharp_M(\omega)$.

The characterization of the space $V^\sharp_M(\omega)$ depends on the geometry of the shell middle surface, shape of the lateral boundary, and type of lateral boundary condition.

If the shell is a totally clamped elliptic shell, by the inequality (6.6.1), it is easily determined that

$$V^\sharp_M(\omega) = \underset{\sim}{H}^1_0(\omega) \times L_2(\omega),$$

and, as we have shown, the mild condition (6.2.9) is enough to guarantee the existence of $\underset{\approx}{\kappa} \in V_0$ such that (6.7.5) holds.

If the shell is a stiff hyperbolic shell, it was shown in [44], see also [18], that

$$V^\sharp_M(\omega) = \text{ a closed subspace of } \quad \underset{\sim}{L}_2(\omega) \times H^{-1}(\omega),$$

therefore, the existence of $\underset{\approx}{\kappa} \in V_0$ is guaranteed if

$$p^\alpha_e + q^\alpha_a - 2b^\alpha_\gamma p^\gamma_o \in L_2(\omega), \quad p^\alpha_o|_\alpha + p^3_e + q^3_a \in H^1_0(\omega), \quad \text{and} \quad p^\alpha_0 = 0. \qquad (6.7.14)$$

If the shell is a stiff parabolic shell, it was shown in [44], see also [18], that

$$V^\sharp_M(\omega) = \text{ a closed subspace of } \quad \underset{\sim}{H}^{-1}(\omega) \times H^{-2}(\omega),$$

so, the existence of $\underset{\approx}{\kappa} \in V_0$ is guaranteed if

$$p^\alpha_e + q^\alpha_a - 2b^\alpha_\gamma p^\gamma_o \in H^1_0(\omega), \quad p^\alpha_o|_\alpha + p^3_e + q^3_a \in H^2_0(\omega), \quad \text{and} \quad p^\alpha_0 = 0. \qquad (6.7.15)$$

If the shell is a partially clamped elliptic shell, the space $V_M^\sharp(\omega)$ can be huge and its norm so weak, that the equation (6.7.5) may have no solution even if the loading functions are in $\mathcal{D}(\omega)$, the space of test functions of distribution. In this case, even if the loading functions make the problem solvable, the problem can not afford an infinitesimal smooth perturbation on the loads, see [38].

Since $\underset{\approx}{\gamma}$ defines an isomorphism between $V_M^\sharp(\omega)$ and V_0, so the existence of $\underset{\approx}{\kappa} \in V_0$ means the existence of $(\underset{\sim}{u}, w) \in V_M^\sharp(\omega)$(the element in $V_M^\sharp(\omega)$ must be viewed as an entity, the notation in components might have no usual sense), such that $\underset{\approx}{\kappa} = \underset{\approx}{\gamma}(\underset{\sim}{u}^0, w^0)$. Therefore, the problem (6.7.5) of determining $\underset{\approx}{\kappa} \in V_0$ is equivalent to finding $(\underset{\sim}{u}^0, w^0) \in V_M^\sharp(\omega)$, such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}(\underset{\sim}{u}^0, w^0)\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$

$$= \int_\omega [(p_e^\alpha + q_a^\alpha - 2b_\gamma^\alpha p_o^\gamma)y_\alpha + (p_o^\alpha|_\alpha + p_e^3 + q_a^3)z]\sqrt{a}d\underset{\sim}{x}$$

$$+ \int_{\gamma_T} p_0^\alpha y_\alpha \quad \forall\ (\underset{\sim}{y}, z) \in V_M^\sharp(\omega). \quad (6.7.16)$$

This variational equation is the same as the "generalized membrane shell" model, except that in the formulation of the right hand side we incorporated the odd part of the surface forces. Note that we are not looking for the solution of this generalized membrane shell problem in the space $V_M^\sharp(\omega)$. Our interest is in the existence of $\underset{\approx}{\kappa} \in V_0$.

The case of $\ker \underset{\approx}{\gamma} \neq 0$ can be divided in two different kinds corresponding to $\ker B = 0$ and $\ker B \neq 0$. The first kind is the "second kind generalized membrane shells" of [20] and [18]. The second kind is our second kind membrane–shear shells.

Since there is not an imaginable realistic example of the "second kind generalized membrane shell", we will not discuss it in details here, but just remark that the requirement (6.7.5), which leads to the convergence of the model solution to the 3D solution in the relative energy norm, is not equivalent to the condition imposed in [18]. Our requirement is more restrictive, and might have excluded some situations analyzed there. In other word, some of the "second kind generalized membrane shells" made the equivalent representation ζ^0 of the resultant loading functional $\boldsymbol{f}_0$ belong to $(V^*, W^*]$. The Naghdi-type model is no longer membrane–shear dominated. Their analysis shows that in some weak sense, and in a quotient space, it is still possible to replace the non-membrane dominated problem by a membrane problem.

Examples for the second kind membrane–shear shells include plate stretching and shear dominated plate bending, which have been thoroughly analyzed, and the condition (6.7.5) does not impose a stringent restriction on the loading functions, see [2] and [5]. Another example is the membrane–shear cylindrical shell analyzed in Chapter 2, for which, we have $V = W$, so the condition (6.7.5) is trivially satisfied.

Based on the requirements imposed on the correction functions w_1 and w_2, and the expressions (6.3.31) and (6.3.32), which need to be small, we define w_1 as the solution of the equation

$$\epsilon^2(\nabla w_1, \nabla v)_{\underset{\sim}{L}_2(\omega)} + (w_1, v)_{L_2(\omega)} = (-\frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o, v)_{L_2(\omega)}, \tag{6.7.17}$$

$$w_1 \in H^1_D(\omega), \quad \forall\, v \in H^1_D(\omega),$$

and define

$$w_2 = 0. \tag{6.7.18}$$

The explanation of this choice of the correction functions is similar to that made for totally clamped elliptic shell.

From the definition (6.7.7) of $\underset{\approx}{\gamma}^0$, we know $\underset{\approx}{\gamma}^0 \in \underset{\approx}{L}_2^{\mathrm{sym}}(\omega)$, so we have $a^{\alpha\beta}\gamma^0_{\alpha\beta} \in L_2(\omega)$. By (3.3.38) in Theorem 3.3.6, we have

$$\epsilon \|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o \|_{L_2(\omega)} \to 0 \ \ (\epsilon \to 0). \tag{6.7.19}$$

If we assume

$$\lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o \in [H^1_D(\omega), L_2(\omega)]_{1-\theta,p} \tag{6.7.20}$$

for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or $\theta \in [0,1]$ and $p \in (1,\infty)$, by (3.3.36) in Theorem 3.3.6, we have

$$\begin{aligned} &\epsilon \|w_1\|_{H^1(\omega)} + \| - w_1 - \frac{\lambda}{2\mu+\lambda} a^{\alpha\beta}\gamma^0_{\alpha\beta} + \frac{1}{2\mu+\lambda} p^3_o \|_{L_2(\omega)} \\ &\qquad \lesssim K(\epsilon, \lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o, [L_2(\omega), H^1_D(\omega)]) \lesssim \epsilon^\theta . \end{aligned} \tag{6.7.21}$$

6.7.2 Admissible applied forces

To prove the convergence theorem, in addition to the asymptotic behaviors of the model solution (6.7.9) and (6.7.19), which hinge on the validity of the condition (6.7.5), we also need to bound the term r in the right hand side of the integration identity (6.3.17). Except for some special shells, like plates and spherical shells, the desirable bound can only be obtained under some restrictions on the applied forces on the 3D shell. As a sufficient condition, we adopt the condition of "admissible applied forces" proposed in [18].

We recall that, the 3D shell Ω^ϵ is subjected to body force and surface tractions on the upper and lower surfaces $\Gamma_\pm$, it is clamped along a part of its lateral surface Γ_D, and loaded by a surface force on the remaining part of the lateral face Γ_T. The body force density is $\boldsymbol{q} = q^i \boldsymbol{g}_i$, the upper and lower surface force densities are $\boldsymbol{p}_\pm = p^i_\pm \boldsymbol{g}_i$, and the lateral surface force density is $\boldsymbol{p}_T = p^i_T \boldsymbol{g}_i$. To describe the concept of "admissible applied forces", we consider the work done by these applied forces over an admissible displacement $\boldsymbol{v} = v_i \boldsymbol{g}^i$, which is given by

$$L(\boldsymbol{v}) = \int_{\Omega^\epsilon} q^i v_i + \int_{\Gamma_\pm} p^i v_i + \int_{\Gamma_T} p^i_T v_i. \tag{6.7.22}$$

Let $\boldsymbol{v}^* \in \boldsymbol{H}^1_D(\omega^\epsilon)$ be the displacement solution of the 3D shell problem. By adapting the notation of [18], we denote the actual stress distribution by $F^{ij}_\epsilon = \sigma^{*ij} \in \boldsymbol{L}^{\text{sym}}_2(\omega^\epsilon)$. Therefore,

$$L(\boldsymbol{v}) = \int_{\omega^\epsilon} F^{ij}_\epsilon \chi_{ij}(\boldsymbol{v}) \sqrt{g} d\underset{\sim}{x} dt. \tag{6.7.23}$$

We scale the 3D shell displacement v^{*i} and the stress F^{ij}_ϵ from the coordinate domain ω^ϵ to the fat domain ω^1, and denote the scaled displacement by $\boldsymbol{v}^*(\epsilon)$ and the scaled stress by $F^{ij}(\epsilon)$, by defining

$$\boldsymbol{v}^*(\epsilon)(\underset{\sim}{x}, \frac{t}{\epsilon}) = \boldsymbol{v}^*(\underset{\sim}{x}, t), \quad F^{ij}(\epsilon)(\underset{\sim}{x}, \frac{t}{\epsilon}) = F^{ij}_\epsilon(\underset{\sim}{x}, t), \ \ \forall \ \underset{\sim}{x} \in \omega, \ t \in (-\epsilon, \epsilon). \tag{6.7.24}$$

In [18], the tensor valued function $F^{ij}(\epsilon)$ was directly introduced to reformulate the linear form $L(\boldsymbol{v})$ (6.7.22) in the form (6.7.23), but on the scaled domain ω^1. The connection between the tensor valued function $F^{ij}(\epsilon)$ and the actual stress distribution over the loaded shell is our observation.

The applied forces are called admissible, if

1. $F^{ij}(\epsilon)$ is uniformly bounded in $\boldsymbol{L}_2^{\mathrm{sym}}(\omega^1)$ with respect to ϵ.

2. There exists a tensor field $F^{ij} \in \boldsymbol{L}_2^{\mathrm{sym}}(\omega^1)$, independent of ϵ, such that

$$\lim_{\epsilon \to 0} F^{ij}(\epsilon) = F^{ij} \quad \text{in } \boldsymbol{L}_2^{\mathrm{sym}}(\omega^1), \tag{6.7.25}$$

see page 265 in [18].

Since $F^{ij}(\epsilon)$ is the actual stress distribution scaled to ω^1, this condition also implies the convergence of the scaled strain tensor $\boldsymbol{\chi}^\epsilon(\boldsymbol{v}^*(\epsilon))$ defined in (6.3.34). It seems that this condition has assumed the convergence of the solution of the 3D shell problem when $\epsilon \to 0$. But the question is how to identify the limit F^{ij}. This limit can only be correctly determined by resorting to a lower dimensional shell model. Therefore, the shell theories established under the assumption of “admissible applied forces” is not totally trivial.

From the first condition, we see that the scaled strain $\chi^\epsilon_{ij}(\boldsymbol{v}^*(\epsilon))$ of the shell deformation (see (6.3.33) and (6.3.34) for definition), is uniformly bounded in $\boldsymbol{L}_2^{\mathrm{sym}}(\Omega)$. By the Korn-type inequality on thin shells (6.3.35), we get the following bound on the scaled displacement

$$\epsilon \|\boldsymbol{v}^*(\epsilon)\|_{\boldsymbol{H}^1_D(\omega^1)} \lesssim 1 \tag{6.7.26}$$

Under the second condition, we can extract a weak convergent subsequence from $\{\epsilon\, \boldsymbol{v}^*(\epsilon)\}$ in $\boldsymbol{H}^1_D(\omega^1)$, then find the weak limit and pass to strong convergence, and finally prove the following convergence,

$$\lim_{\epsilon \to 0} \epsilon \|\boldsymbol{v}^*(\epsilon)\|_{\boldsymbol{H}^1_D(\omega^1)} = 0, \tag{6.7.27}$$

see [18] for details. Note that this behavior of the 3D shell solution is compatible with the behavior (6.7.12) of the 2D model solution, yet another evidence for the necessity of the assumption on the admissibility of the applied forces.

By rescaling the convergence (6.7.27) back to the domain ω^ϵ, we will get

$$\epsilon^{1/2}\left(\sum_{\alpha=1}^{2} \|v^*_\alpha\|_{H^1(\omega)\times L_2(-\epsilon,\epsilon)} + \|v^*_3\|_{L_2(\omega^\epsilon)}\right) \lesssim o(1) \tag{6.7.28}$$

This inequality is what we need to prove our theorem.

6.7.3 Convergence theorem

For the general membrane–shear shells, under the condition (6.7.5) assumed on the 2D model problem (6.2.4) and the condition (6.7.25) imposed on the 3D shell problem, we have the convergence theorem:

THEOREM 6.7.1. *Let $\boldsymbol{v}^*$ and $\boldsymbol{\sigma}^*$ be the displacement and stress of the shell determined from the 3D elasticity equations, $\boldsymbol{v}$ the displacement defined through the formulae (6.3.13)* *in terms of the 2D model solution* $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ *and the correction functions* w_1, w_2 *defined in (6.7.17) and (6.7.18), and* $\boldsymbol{\sigma}$ *the stress field defined by (6.3.7). We have the convergence*

$$\lim_{\epsilon\to 0} \frac{\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0. \qquad (6.7.29)$$

Proof. Except for the different way to bound the term r in the identity (6.3.17), the proof is otherwise the same as that of the Theorems 6.6.1 and 5.5.1. The proof is based on the identity (6.3.17), the inequality (6.7.28), the two convergences (6.7.9) and (6.7.19), the inequality (6.3.16) to bound $\sigma_2^{\alpha\beta}$, and the expressions (6.3.28), (6.3.29) and (6.3.30) for the constitutive residual ϱ_{ij}. Again, for brevity, the norm $\|\cdot\|_{L_2(\omega^\epsilon)}$ will be simply denoted by $\|\cdot\|$. Any function defined on ω will be viewed as a function, constant in t, defined on ω^ϵ.

First, we establish the lower bound for the strain energy engendered by the displacement $\boldsymbol{v}$. By the convergence (6.7.9), we have

$$\epsilon\,\|\underset{\approx}{\rho}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim o(1), \quad \|\underset{\approx}{\gamma}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim o(1), \quad \|\underset{\sim}{\tau}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim o(1). \quad (6.7.30)$$

Since $\underset{\approx}{\gamma}^0$ and $\underset{\sim}{\tau}^0$ can not be zero at the same time (otherwise $\boldsymbol{f}_0 = 0$), we have

$$\|\underset{\approx}{\gamma}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \simeq \|\underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \simeq 1. \qquad (6.7.31)$$

By the equivalence (6.2.8), we have

$$\epsilon\,\|(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)\|_{\underset{\sim}{H}^1(\omega)\times\underset{\sim}{H}^1(\omega)\times H^1(\omega)} \lesssim o(1). \qquad (6.7.32)$$

The convergence (6.7.19) shows

$$\epsilon\,\|w_1\|_{H^1(\omega)} \lesssim o(1) \quad \text{and} \quad \|w_1\|_{L_2(\omega)} \simeq \|\lambda a^{\alpha\beta}\gamma^0_{\alpha\beta} - p^3_o\|_{L_2(\omega)}. \tag{6.7.33}$$

Recalling the expression (6.3.24), we have

$$\chi_{\alpha\beta}(\boldsymbol{v}) = \gamma^\epsilon_{\alpha\beta} + t\rho^\epsilon_{\alpha\beta} - t(b^\lambda_\alpha\gamma^\epsilon_{\lambda\beta} + b^\lambda_\beta\gamma^\epsilon_{\lambda\alpha}) + t(tc_{\alpha\beta} - b_{\alpha\beta})w_1 - \frac{1}{2}t^2(b^\gamma_\alpha\theta^\epsilon_{\gamma|\beta} + b^\gamma_\beta\theta^\epsilon_{\gamma|\alpha})$$

and

$$\chi_{\alpha 3}(\boldsymbol{v}) = \frac{1}{2}\tau^\epsilon_\alpha + \frac{1}{2}t\partial_\alpha w_1,$$

in which, by the estimates (6.7.30), (6.7.31), (6.7.32), and (6.7.33), the terms $\gamma^\epsilon_{\alpha\beta}$ and τ^ϵ_α dominate respectively. Summerizing these estimates, we get

$$\sum_{\alpha,\beta=1}^{2} \|\chi_{\alpha\beta}(\boldsymbol{v})\|^2 + \sum_{\alpha=1}^{2} \|\chi_{3\alpha}(\boldsymbol{v})\|^2 \gtrsim \epsilon[\|\underset{\approx}{\gamma}^\epsilon\|^2_{\underset{\approx}{L}^{\text{sym}}_2(\omega)} + \|\underset{\sim}{\tau}^\epsilon\|^2_{\underset{\sim}{L}_2(\omega)}] \gtrsim \epsilon\,.$$

Therefore,

$$\|\boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon} \gtrsim \epsilon\,. \tag{6.7.34}$$

We then derive the upper bound on $\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|^2_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon}$. From the identity (6.3.17), we have

$$\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|^2_{E^\epsilon} + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|^2_{E^\epsilon} \lesssim \sum_{i,j=1}^{3} \|\varrho_{ij}\|^2 + |r| \tag{6.7.35}$$

From the expression (6.3.21) of r, we see

$$|r| \lesssim \epsilon \sum_{\alpha,\beta=1}^{2} (\|\sigma_0^{\alpha\beta}\| + \|\sigma_2^{\alpha\beta}\|)[\sum_{\alpha=1}^{2} \|v_\alpha^*\|_{H^1(\omega)\times L_2(-\epsilon,\epsilon)} + \|v_3^*\|]$$

$$+ \epsilon^{5/2} \sum_{\alpha,\beta=1}^{2} (\|\sigma_0^{\alpha\beta}\| + \|\sigma_2^{\alpha\beta}\|)[\|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1(\omega)} + \|w_1\|_{L_2(\omega)}]. \quad (6.7.36)$$

From the equations (6.3.1) and (6.7.8), we have

$$\sigma_0^{\alpha\beta} = \frac{2}{3} H \,\epsilon^2 \,\sigma_1^{\alpha\beta} + a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda} p_o^3 a^{\alpha\beta},$$

$$\sigma_1^{\alpha\beta} = a^{\alpha\beta\lambda\gamma}\rho_{\lambda\gamma}^{\epsilon} + \frac{\lambda}{2\mu+\lambda}(p_e^3 + 2Hp_o^3)a^{\alpha\beta},$$

$$\sigma_0^{3\alpha} = \frac{5}{4}[\mu a^{\alpha\beta}\tau_\beta^{\epsilon} - p_o^{\alpha}] = \frac{5}{4}[\mu a^{\alpha\beta}(\tau_\beta^{\epsilon} - \tau_\beta^{0})],$$

and so, the estimates

$$\epsilon^2 \,\|\sigma_1^{\alpha\beta}\|^2 \lesssim o(\epsilon), \quad \|\sigma_0^{\alpha\beta}\|^2 \lesssim \epsilon, \quad \|\sigma_0^{3\alpha}\|^2 \lesssim o(\epsilon). \quad (6.7.37)$$

By the estimate (6.3.16), we have

$$\|\sigma_2^{\alpha\beta}\|^2 \lesssim o(\epsilon). \quad (6.7.38)$$

Combining (6.7.37) and (6.7.38), we see

$$\sum_{\alpha,\beta=1}^{2} (\|\sigma_0^{\alpha\beta}\| + \|\sigma_2^{\alpha\beta}\|) \lesssim O(\epsilon^{1/2}). \quad (6.7.39)$$

Together with the inequality (6.7.28), we get the upper bound on the first term in the right hand side of (6.7.36):

$$\epsilon \sum_{\alpha,\beta=1}^{2} (\|\sigma_0^{\alpha\beta}\| + \|\sigma_2^{\alpha\beta}\|)[\sum_{\alpha=1}^{2} \|v_\alpha^*\|_{H^1(\omega)\times L_2(-\epsilon,\epsilon)} + \|v_3^*\|] \lesssim o(\epsilon). \tag{6.7.40}$$

Using (6.7.32), (6.7.33) and (6.7.39), we get the bound on the second term:

$$\epsilon^{5/2} \sum_{\alpha,\beta=1}^{2} (\|\sigma_0^{\alpha\beta}\| + \|\sigma_2^{\alpha\beta}\|)[\|\underset{\sim}{\theta}\|_{\underset{\sim}{H}^1(\omega)} + \|w_1\|_{L_2(\omega)}] \lesssim o(\epsilon^2). \tag{6.7.41}$$

Therefore, we obtain

$$|r| \lesssim o(\epsilon) \tag{6.7.42}$$

The proof of

$$\sum_{i,j=1}^{3} \|\varrho_{ij}\|^2 \lesssim o(\epsilon)$$

is a verbatim repetition of the relevant part in the proof of Theorem 5.5.1. By (6.7.35), we get

$$\|\boldsymbol{\sigma}^* - \boldsymbol{\sigma}\|_{E^\epsilon}^2 + \|\boldsymbol{\chi}(\boldsymbol{v}^*) - \boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}^2 \lesssim o(\epsilon).$$

The conclusion of the theorem follows from this inequality and the lower bound (6.7.34).

□

To get a convergence rate, we need to use the asymptotic behaviors (6.7.11) and (6.7.21) of the model solution, whose validity depends on the assumptions (6.7.10) and (6.7.20), and a more strict requirement on the applied forces on the 3D shell. Otherwise, the statement of the theorem on the convergence rate is the same as Theorem 6.6.2, and the proof would be a modification of that of Theorem 6.7.1.

The conclusion of this theorem seems stronger than other theories for the general membrane–shear shells.

Chapter 7

Discussions and justifications of other linear shell models

In this final chapter, we briefly discuss the justifications of some other linear shell models based on the convergence theorems we proved for the model (6.2.4). The discussion is in the context of Chapter 6. All these models can be viewed as variants of the general shell model (6.2.4). We recall that the solution of the general shell model (6.2.4) was denoted by $(\underset{\sim}{\theta}^{\epsilon}, \underset{\sim}{u}^{\epsilon}, w^{\epsilon})$. In terms of this model solution and the transverse deflection correction functions w_1 and w_2, we defined an admissible displacement field $\boldsymbol{v}$ by the formulae (6.3.13). The convergence and convergence rate in the relative energy norm of $\boldsymbol{v}$ toward the 3D displacement solution $\boldsymbol{v}^*$ were proved.

For each variant of the general shell model, we will re-define displacement functions $(\underset{\sim}{\bar{\theta}}^{\epsilon}, \underset{\sim}{\bar{u}}^{\epsilon}, \bar{w}^{\epsilon}) \in H$ from its solution. The the correction functions w_1 and w_2 will be defined either by (6.5.8) and (6.5.9) or by (6.7.17) and (6.7.18), depending on whether the model problem is flexural or of membrane–shear. In terms of $(\underset{\sim}{\bar{\theta}}^{\epsilon}, \underset{\sim}{\bar{u}}^{\epsilon}, \bar{w}^{\epsilon})$ and w_1 and w_2, we define an admissible displacement field $\bar{\boldsymbol{v}}$ by the formulae (6.3.13). We will use the notations

$$\bar{\gamma}^{\epsilon}_{\alpha\beta} = \gamma_{\alpha\beta}(\underset{\sim}{\bar{u}}^{\epsilon}, \bar{w}^{\epsilon}), \quad \bar{\rho}^{\epsilon}_{\alpha\beta} = \rho_{\alpha\beta}(\underset{\sim}{\bar{\theta}}^{\epsilon}, \underset{\sim}{\bar{u}}^{\epsilon}, \bar{w}^{\epsilon}), \quad \bar{\tau}^{\epsilon}_{\alpha} = \tau_{\alpha}(\underset{\sim}{\bar{\theta}}^{\epsilon}, \underset{\sim}{\bar{u}}^{\epsilon}, \bar{w}^{\epsilon}),$$

which give the membrane, flexural, and transverse shear strains (6.2.3) engendered by $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon)$. By the formulae (6.3.24), we can easily get the expression for $\boldsymbol{\chi}(\boldsymbol{v}) - \boldsymbol{\chi}(\bar{\boldsymbol{v}})$, which is the difference between the 3D strain tensors engendered by $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$:

$$\begin{aligned}\chi_{\alpha\beta}(\boldsymbol{v}) - \chi_{\alpha\beta}(\bar{\boldsymbol{v}}) &= \gamma^\epsilon_{\alpha\beta} - \bar{\gamma}^\epsilon_{\alpha\beta} + t(\rho^\epsilon_{\alpha\beta} - \bar{\rho}^\epsilon_{\alpha\beta}) - t[b^\lambda_\alpha(\gamma^\epsilon_{\lambda\beta} - \bar{\gamma}^\epsilon_{\lambda\beta}) + b^\lambda_\beta(\gamma^\epsilon_{\lambda\alpha} - \bar{\gamma}^\epsilon_{\lambda\alpha})] \\ &\quad -\frac{1}{2}t^2[b^\gamma_\alpha(\theta^\epsilon_{\gamma|\beta} - \bar{\theta}^\epsilon_{\gamma|\beta}) + b^\gamma_\beta(\theta^\epsilon_{\gamma|\alpha} - \bar{\theta}^\epsilon_{\gamma|\alpha})], \\ \chi_{\alpha 3}(\boldsymbol{v}) - \chi_{\alpha 3}(\bar{\boldsymbol{v}}) &= \chi_{3\alpha}(\boldsymbol{v}) - \chi_{3\alpha}(\bar{\boldsymbol{v}}) = \frac{1}{2}(\tau^\epsilon_\alpha - \bar{\tau}^\epsilon_\alpha), \quad \chi_{33}(\boldsymbol{v}) - \chi_{33}(\bar{\boldsymbol{v}}) = 0. \end{aligned} \tag{7.0.1}$$

The variant of the model will be justified by proving the convergence rate

$$\frac{\|\boldsymbol{\chi}(\boldsymbol{v}) - \boldsymbol{\chi}(\bar{\boldsymbol{v}})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} \lesssim \epsilon^\theta, \tag{7.0.2}$$

or the convergence

$$\lim_{\epsilon \to 0} \frac{\|\boldsymbol{\chi}(\boldsymbol{v}) - \boldsymbol{\chi}(\bar{\boldsymbol{v}})\|_{E^\epsilon}}{\|\boldsymbol{\chi}(\boldsymbol{v})\|_{E^\epsilon}} = 0, \tag{7.0.3}$$

which together with the theorems of Chapter 6 give convergence rate or convergence of the solution of the variant of the model to the 3D solution in the relative energy norm.

7.1 Negligibility of the higher order term in the loading functional

We first show that the higher order term $\epsilon^2 \boldsymbol{f}_1$ in the resultant loading functional of the model (6.2.4) is negligible. Let's just retain the leading term $\boldsymbol{f}_0$ in the loading functional, and denote the solution by $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon) \in H$. For flexural shells, by

Theorem 3.3.2 and Theorem 3.3.3, we can prove the estimates

$$\|\bar{\underset{\approx}{\rho}}^{\epsilon} - \underset{\approx}{\rho}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\,\|\bar{\underset{\approx}{\gamma}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\,\|\bar{\underset{\sim}{\tau}}^{\epsilon}\|_{\underset{\sim}{L}_2(\omega)} \lesssim K(\epsilon, \xi^0, [W^*, V^*]) \lesssim \epsilon^{\theta}, \tag{7.1.1}$$

if the regularity condition (6.5.5) is satisfied by ξ^0. If we only have $\xi^0 \in W^*$, then

$$\|\bar{\underset{\approx}{\rho}}^{\epsilon} - \underset{\approx}{\rho}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\,\|\bar{\underset{\approx}{\gamma}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\,\|\bar{\underset{\sim}{\tau}}^{\epsilon}\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{7.1.2}$$

For membrane–shear shells, by Theorem 3.3.4 and Theorem 3.3.5 we have

$$\epsilon\,\|\bar{\underset{\approx}{\rho}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^{\epsilon} - \underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\sim}{\tau}}^{\epsilon} - \underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}_2(\omega)} \lesssim K(\epsilon, (\underset{\approx}{\gamma}^{0}, \underset{\sim}{\tau}^{0}), [V, W]) \lesssim \epsilon^{\theta}, \tag{7.1.3}$$

if the condition (6.6.11) or (6.7.10) is satisfied by $(\underset{\approx}{\gamma}^{0}, \underset{\sim}{\tau}^{0})$. If we only have $(\underset{\approx}{\gamma}^{0}, \underset{\sim}{\tau}^{0}) \in V$, the convergence

$$\epsilon\,\|\bar{\underset{\approx}{\rho}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^{\epsilon} - \underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\sim}{\tau}}^{\epsilon} - \underset{\sim}{\tau}^{0}\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0) \tag{7.1.4}$$

holds. Here, $\underset{\approx}{\rho}^{0}$, $\underset{\approx}{\gamma}^{0}$, $\underset{\sim}{\tau}^{0}$, and ξ^0 are what were defined in Chapter 6. Combining (7.1.1) with (6.5.6) together with the lower bound (6.5.18), under the condition of Theorem 6.5.2, we will get the convergence rate (7.0.2) for flexural shells. Combining (7.1.2) with (6.5.4), under the condition of Theorem 6.5.1, we get the convergence (7.0.3) for flexural shells. Similarly, under the condition of Theorem 6.6.1 or Theorem 6.7.1, the estimate (7.1.3), the estimate (6.6.12) or (6.7.11), and the lower bound (6.7.34) together lead to the convergence rate for membrane–shear shells, and under the condition of Theorem 6.6.1

or Theorem 6.7.1, the estimate (7.1.4) and the estimate (6.6.10) or (6.7.9) give the convergence. Therefore, we just need to keep the leading term $\boldsymbol{f}_0$ in the resultant loading functional. Cutting-off the higher order term $\epsilon^2\,\boldsymbol{f}_1$ will not affect the convergence property of the model solution to the 3D solution in the relative energy norm.

It should be noted that the higher order term $\epsilon^2\,\boldsymbol{f}_1$ is also negligible in other norms for flexural shells and stiff membrane–shear shells, but in the case of the second kind membrane–shear shells, if $\boldsymbol{f}_0|_{\ker B} = 0$ but $\boldsymbol{f}_1|_{\ker B} \neq 0$, although the contribution of the higher order term $\epsilon^2\,\boldsymbol{f}_1$ is negligible in the energy norm, but it might be significant in other norms. To see this, we use $(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon)$ to denote the solution of the model (6.2.4) in which the loading functional is replaced by $\epsilon^2\,\boldsymbol{f}_1$. The above analysis has already shown the negligibility of $(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon)$ in the relative energy norm. On the other hand, by our analysis of the flexural shell, see (6.5.4), we have the convergence

$$\begin{aligned}
&\|\underset{\approx}{\rho}(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon) - \underset{\approx}{\rho}{}_1^0\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)} \\
&\qquad + \epsilon^{-1}\,\|\underset{\approx}{\gamma}(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon)\|_{\underset{\approx}{L}{}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\,\|\underset{\sim}{\tau}(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon)\|_{\underset{\sim}{L}{}_2(\omega)} \to 0 \quad (\epsilon \to 0),
\end{aligned}$$

in which $\underset{\approx}{\rho}{}_1^0 \neq 0$ is defined by the limiting flexural model (6.5.3) with $\boldsymbol{F}_0$ replaced by $\boldsymbol{f}_1$. So, $(\underset{\sim}{\theta}{}_1^\epsilon, \underset{\sim}{u}{}_1^\epsilon, w_1^\epsilon)$ does not converge to zero in, say, the L_2 norm. Therefore, in this special case, we can not determine the convergence of the model (6.2.4), either with or without $\epsilon^2\,\boldsymbol{f}_1$, in norms other than the relative energy norm. For plates, asymptotic analysis shows that the higher order term helps in this case. In the following discussion, we will discard $\epsilon^2\,\boldsymbol{f}_1$. When we mention the model (6.2.4), the loading functional is understood as $\boldsymbol{f}_0$.

7.2 The Naghdi model

The Naghdi model can be obtained by replacing the flexural strain operator $\underset{\approx}{\rho}$ in the model (6.2.4) with $\underset{\approx}{\rho}^N$. The model reads: Find $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \in H$, such that

$$\frac{1}{3}\epsilon^2 \int_\omega a^{\alpha\beta\lambda\gamma} \rho^N_{\lambda\gamma}(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon) \rho^N_{\alpha\beta}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$+ \int_\omega a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\underset{\sim}{u}^\epsilon, w^\epsilon)\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} + \frac{5}{6}\mu \int_\omega a^{\alpha\beta}\tau_\beta(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)\tau_\alpha(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$
$$= \langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle \quad \forall\ (\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in H, \quad (7.2.1)$$

in which $\boldsymbol{f}_0$ is what was defined by (6.2.5), and

$$\rho^N_{\alpha\beta}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) - \frac{1}{2}(b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha u_{\lambda|\beta}) + c_{\alpha\beta}w.$$

Let's define $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon) = (\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$. This model can be fitted in the abstract problem of Chapter 3, and classified in the same way in which we classified the model (6.2.4). If in (7.1.1), (7.1.2), (7.1.3), or (7.1.4), $\underset{\approx}{\bar{\rho}}^\epsilon$ is repalced by $\underset{\approx}{\rho}^N(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon)$, these estimates hold under exactly the same conditions. From the relation

$$\rho_{\alpha\beta} = \rho^N_{\alpha\beta} + b^\lambda_\alpha\gamma_{\lambda\beta} + b^\gamma_\beta\gamma_{\gamma\alpha},$$

it is easy to see that the estimates (7.1.1), (7.1.2), (7.1.3), or (7.1.4) themselves hold. By defining the corrections w_1 and w_2 in the same way as of Chapter 6, we can define an admissible displacement field $\bar{\boldsymbol{v}}$ by the formulae (6.3.13). The convergence properties we established in the last chapter for the model (6.2.4) all apply to this Naghdi model.

In the Naghdi model (7.2.1), if the correction factor 5/6 of the transverse shear term and the factor 5/6 in the first term of the expression of $\boldsymbol{f}_0$ (6.2.5) are replaced by 1 simultaneously, all the convergence theorems are still true.

7.3 The Koiter model and the Budianski–Sanders model

The Koiter model is defined as the restriction of the Naghdi model (7.2.1) on the subspace $H^K = \{(\underset{\sim}{\theta}, \underset{\sim}{u}, w) : (\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H; \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = 0\}$ of $H = \underset{\sim}{H}^1_D(\omega) \times \underset{\sim}{H}^1_D(\omega) \times H^1_D(\omega)$. The constraint $\underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w) = 0$ is equivalent to $\theta_\alpha = -\partial_\alpha w - b^\lambda_\alpha u_\lambda$. So, it removed the independent variable $\underset{\sim}{\theta}$. From this constraint we also see that $\partial_\alpha w \in H^1_D(\omega)$. Therefore, the space $H^K = \underset{\sim}{H}^1_D(\omega) \times H^2_D(\omega)$. Constrained on H^K, the model (7.2.1) becomes: Find $(\underset{\sim}{u}^\epsilon, w^\epsilon) \in H^K$, such that

$$\frac{1}{3}\epsilon^2 \int_\omega a^{\alpha\beta\lambda\gamma} \rho^K_{\lambda\gamma}(\underset{\sim}{u}^\epsilon, w^\epsilon) \rho^K_{\alpha\beta}(\underset{\sim}{y}, z) \sqrt{a} d\underset{\sim}{x}$$
$$+ \int_\omega a^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma}(\underset{\sim}{u}^\epsilon, w^\epsilon) \gamma_{\alpha\beta}(\underset{\sim}{y}, z) \sqrt{a} d\underset{\sim}{x} = \langle \boldsymbol{f}^K, (\underset{\sim}{y}, z) \rangle \quad \forall\, (\underset{\sim}{y}, z) \in H^K. \quad (7.3.1)$$

The operator $\underset{\approx}{\rho}^K$ is the restriction of the operator $\underset{\approx}{\rho}^N$ on H^K:

$$\rho^K_{\alpha\beta}(\underset{\sim}{u}, w) = -w|_{\alpha\beta} - b^\lambda_{\alpha|\beta} u_\lambda - (b^\lambda_\alpha u_{\lambda|\beta} + b^\lambda_\beta u_{\lambda|\alpha}) + c_{\alpha\beta} w,$$

here, $w|_{\alpha\beta} = \partial^2_{\alpha\beta} w - \Gamma^\gamma_{\alpha\beta} \partial_\gamma w$, and the resultant loading functional is the restriction of $\boldsymbol{f}_0$ on H^K:

$$\langle \boldsymbol{f}^K, (\underset{\sim}{y}, z) \rangle = -\frac{\lambda}{2\mu + \lambda} \int_\omega p^3_o a^{\alpha\beta} \gamma_{\alpha\beta}(\underset{\sim}{y}, z) \sqrt{a} d\underset{\sim}{x}$$

$$+\int_\omega [(p_e^\alpha + q_a^\alpha - 2b_\gamma^\alpha p_o^\gamma)y_\alpha + (p_o^\alpha|_\alpha + p_e^3 + q_a^3)z]\sqrt{a}d\underset{\sim}{x} + \int_{\gamma_T} p_0^\alpha y_\alpha,$$

The well posedness of this Koiter model easily follows from that of the Naghdi model if we assume $\boldsymbol{f}^K$ is in the dual space of H^K. Based on the model solution $(\underset{\sim}{u}^\epsilon, w^\epsilon) \in H^K$, we can define $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon) \in H$ by setting $\bar{\theta}_\alpha^\epsilon = -\partial_\alpha w^\epsilon - b_\alpha^\lambda u_\lambda^\epsilon$, $\underset{\sim}{\bar{u}}^\epsilon = \underset{\sim}{u}^\epsilon$, and $\bar{w}^\epsilon = w^\epsilon$. By defining the transverse corrections w_1 and w_2 in exactly the same way as of Chapter 6, we can construct the admissible displacement field $\bar{\boldsymbol{v}}$ by the formula (6.3.13). The Koiter model can also be fitted in the abstract problem of Chapter 3 by properly defining operators and spaces. The problem can be accordingly classified as a flexural shell or a membrane shell (no shear). For flexural shells, by the same scaling on the loading functions, the estimate

$$\|\underset{\approx}{\bar{\rho}}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\|\underset{\approx}{\bar{\gamma}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^\theta, \quad \underset{\sim}{\bar{\tau}}^\epsilon = 0, \tag{7.3.2}$$

or the convergence

$$\|\underset{\approx}{\bar{\rho}}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \epsilon^{-1}\|\underset{\approx}{\bar{\gamma}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \to 0 \ \ (\epsilon \to 0), \quad \underset{\sim}{\bar{\tau}}^\epsilon = 0 \tag{7.3.3}$$

can be proved, depending on the "regularity" of the Lagrange multiplier associated with a limiting problem which is slightly different from (6.5.3). Here, $\underset{\approx}{\rho}^0$ is the same as what was defined in (6.5.3). The value of θ might not be the same as what was defined in (6.5.5). If different, we take the least one to determine the model convergence rate. The estimate (7.3.2) and the estimate (6.5.6) together will give a convergence rate of the form

(7.0.2). The convergence (7.3.3) together with (6.5.4) lead to a convergence of the form (7.0.3).

For membrane shells, the estimate

$$\epsilon \, \|\bar{\underset{\approx}{\rho}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^{\epsilon} - \underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \lesssim \epsilon^{\theta}, \qquad \bar{\underset{\sim}{\tau}}^{\epsilon} = 0, \tag{7.3.4}$$

or the convergence

$$\epsilon \, \|\bar{\underset{\approx}{\rho}}^{\epsilon}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^{\epsilon} - \underset{\approx}{\gamma}^{0}\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} \to 0 \quad (\epsilon \to 0), \quad \bar{\underset{\sim}{\tau}}^{\epsilon} = 0. \tag{7.3.5}$$

can be proved depending on the "regularity" of $\underset{\approx}{\gamma}^{0}$. Here, $\underset{\approx}{\gamma}^{0}$ is what was defined in (6.6.8) or (6.7.7). Again, the value of θ might be different from that defined in (6.6.11) or (6.7.10). For example, for a totally clamped elliptic shell, our estimate of the value of θ in (6.6.11) is 1/6, while from (6.6.28), we know that the value of θ in (7.3.4) should be 1/5. Note that if $\tau_{\alpha}^{0} = \frac{1}{\mu} a_{\alpha\beta} p_{o}^{\beta} \neq 0$, these estimates are essentially different from (6.6.12) and (6.6.10), or (6.7.11) and (6.7.9). In this case, the difference (7.0.1) can not be small, so the model can not be justified. Actually, the Koiter model diverges. If, say, $p_{o}^{\alpha} = 0$, then the difference (7.0.1) is small, and the Koiter model can therefore be justified, and for totally clamped elliptic shells, by (6.6.28), we know that the convergence rate of the Koiter model is $O(\epsilon^{1/5})$, which is the same as that of the model (6.2.4).

The Budianski–Sanders model is a variant of the Koiter model. The only difference is in the flexural strain operator. If in the Koiter model (7.3.1), $\underset{\approx}{\rho}^{K}$ is replaced by

$\underset{\approx}{\rho}^{BS}$ that is defined by

$$\rho^{BS}_{\alpha\beta} = \rho^{K}_{\alpha\beta} + \frac{1}{2}(b^{\lambda}_{\alpha}\gamma_{\lambda\beta} + b^{\gamma}_{\beta}\gamma_{\gamma\alpha}),$$

we will get the Budianski–Sanders model. By using the theory of Chapter 3 and the above relation, we can easily get a convergence or an estimates of the form (7.3.2), (7.3.3), (7.3.4), or (7.3.5). So the convergence property of the Budianski–Sanders model is the same as that of Koiter's.

7.4 The limiting models

For flexural shells, the limiting model is the variational problem (6.5.3) which is defined on $\ker B$: Find $(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0) \in \ker B$, such that

$$\frac{1}{3}\int_\omega a^{\alpha\beta\lambda\gamma}\rho_{\lambda\gamma}(\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)\rho_{\alpha\beta}(\underset{\sim}{\phi}, \underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} = \langle \boldsymbol{F}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle_{H^*\times H}$$

$$\forall(\underset{\sim}{\phi}, \underset{\sim}{y}, z) \in \ker B. \quad (7.4.1)$$

We let $(\underset{\sim}{\bar\theta}^\epsilon, \underset{\sim}{\bar u}^\epsilon, \bar w^\epsilon) = (\underset{\sim}{\theta}^0, \underset{\sim}{u}^0, w^0)$, and so we have

$$\|\underset{\approx}{\bar\rho}^\epsilon - \underset{\approx}{\rho}^0\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} + \epsilon^{-1}\|\underset{\approx}{\bar\gamma}^\epsilon\|_{\underset{\approx}{L}^{\mathrm{sym}}_2(\omega)} + \epsilon^{-1}\|\underset{\sim}{\bar\tau}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} = 0.$$

We define the corrections w_1 and w_2 with (6.5.8) and (6.5.9). The above equation together with the estimate (6.5.6) or the convergence (6.5.4) prove the convergence rate of the form (7.0.2) under the condition of Theorem 6.5.2, or the convergence of the form (7.0.3) under the condition of Theorem 6.5.1.

For totally clamped elliptic shells, we assume that the shell data satisfy the smoothness assumption of Section 6.6.4. The limiting model reads: Find $(\underset{\sim}{u}^0, w^0) \in H_0^1(\omega) \times L_2(\omega)$ such that

$$\int_\omega a^{\alpha\beta\lambda\gamma}\gamma_{\lambda\gamma}(\underset{\sim}{u}^0, w^0)\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x} = -\frac{\lambda}{2\mu+\lambda}\int_\omega p_o^3 a^{\alpha\beta}\gamma_{\alpha\beta}(\underset{\sim}{y}, z)\sqrt{a}d\underset{\sim}{x}$$

$$+\int_\omega [(p_e^\alpha + q_a^\alpha - 2b_\gamma^\alpha p_o^\gamma)y_\alpha + (p_o^\alpha|_\alpha + p_e^3 + q_a^3)z]\sqrt{a}d\underset{\sim}{x}$$

$$\forall\ (\underset{\sim}{y}, z) \in \underset{\sim}{H}_0^1(\omega) \times L_2(\omega). \quad (7.4.2)$$

By construction, we have already shown in Section 6.6.4 that there exists a $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$, with $\underset{\sim}{u} = \underset{\sim}{u}^0$, such that

$$\|[\underset{\approx}{\gamma}^0 - \underset{\approx}{\gamma}(\underset{\sim}{u}, w), \underset{\sim}{\tau}^0 - \underset{\sim}{\tau}(\underset{\sim}{\theta}, \underset{\sim}{u}, w)]\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)\times\underset{\sim}{L}_2(\omega)} + \epsilon\,\|(\underset{\sim}{\theta}, \underset{\sim}{u}, w)\|_H \lesssim \epsilon^{1/6}. \quad (7.4.3)$$

Therefore, we can take $(\bar{\underset{\sim}{\theta}}^\epsilon, \bar{\underset{\sim}{u}}^\epsilon, \bar{w}^\epsilon) = (\underset{\sim}{\theta}, \underset{\sim}{u}^0, w)$. From (7.4.3), we have

$$\epsilon\,\|\bar{\underset{\approx}{\rho}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\sim}{\tau}}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^{1/6}.$$

We define w_1 and w_2 by (6.6.13) and (6.6.14). This estimate and the estimate (6.6.12) (in which $\theta = 1/6$) together lead to an estimate of the form (7.0.2) in which $\theta = 1/6$. The convergence rate of $\bar{\boldsymbol{v}}$ to the 3D solution in the relative energy norm then is $O(\epsilon^{1/6})$.

If $\tau_\alpha^0 = \dfrac{1}{\mu} a_{\alpha\beta} p_o^\beta = 0$, we can take $(\underset{\sim}{\theta}, \underset{\sim}{u}, w) \in H$ as defined on page 203, and define $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon)$ in the same way, we then have

$$\epsilon \|\underset{\approx}{\bar{\rho}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\approx}{\bar{\gamma}}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\underset{\sim}{\bar{\tau}}^\epsilon - \underset{\sim}{\tau}^0\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^{1/5}\,.$$

The convergence rate of $\bar{\boldsymbol{v}}$ to the 3D solution in the relative energy norm then is $O(\epsilon^{1/5})$ Note that without $\underset{\sim}{\bar{u}}^\epsilon = \underset{\sim}{u}^0$, we can not say the above argument furnishes a justification for the limiting model (7.4.2).

For other membrane–shear shells, the limiting model is defined by (6.7.5), the form of which is the same as the limiting model for totally clamped elliptic shells, but the model is defined on the space $V_M^\sharp(\omega)$. It is easy to show that there exists $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon) \in H$ such that an estimate of the form (7.1.3) or (7.1.4) hold, but it seems that the best way to find such $(\underset{\sim}{\bar{\theta}}^\epsilon, \underset{\sim}{\bar{u}}^\epsilon, \bar{w}^\epsilon)$ might be solving the model (6.2.4). The limiting model is hardly useful.

7.5 About the loading assumption

In our analysis of the shell models, we have assumed that the components of the odd part of surface forces p_o^i, the components of the weighted even part of surface forces p_e^i, the components of the body force q_a^i, and the coefficients of the rescaled lateral surface force components p_0^i, p_1^i, and p_2^i are all independent of ϵ. This assumption is different from the assumption assumed in asymptotic theories, see [18]. In this section, we briefly discuss the justification of the general shell model (6.2.4) under the loading assumption of asymptotic theories.

With a slight abuse of notations, we now use p_o^i to denote the components of the weighted odd part of the surface forces in this section, i.e., $p_o^i = (\tilde{p}_+^i - \tilde{p}_-^i)/2\,\epsilon$. The meanings of p_e^i, q_a^i, and p_0^i, p_1^i, and p_2^i are the same as before. The new loading assumption then is that p_o^i, p_e^i, q_a^i, and p_0^i, p_1^i, and p_2^i are all independent of ϵ.

With the changed meaning of p_o^i, the form of the resultant loading functional in the model (6.2.4) will be changed. By replacing p_o^i with $\epsilon\, p_o^i$ in (6.2.5) and (6.2.6), the loading functional will be changed to $\boldsymbol{f}_0 + \epsilon\, \boldsymbol{f}_1 + \epsilon^2\, \boldsymbol{f}_2 + \epsilon^3\, \boldsymbol{f}_3$, in which, the leading term is given by

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, \underset{\sim}{y}, z)\rangle = \int_\omega [(p_e^\alpha + q_a^\alpha) y_\alpha + (p_e^3 + q_a^3) z] \sqrt{a}\, d\underset{\sim}{x} + \int_{\gamma_T} p_0^\alpha y_\alpha.$$

This functional is roughly the same as the loading functional obtained by asymptotic analysis. By using (3.4.9) and (3.4.10) we can get the asymptotic behavior of the model solution $(\underset{\sim}{\theta}^\epsilon, \underset{\sim}{u}^\epsilon, w^\epsilon)$ when $\epsilon \to 0$. The justification of the model is otherwise the same as what we did in Chapter 6. For flexural shells, there is nothing new. For membrane–shear shells, in the expression of $\zeta^0 = (\underset{\approx}{\gamma}^0, \underset{\sim}{\tau}^0)$, the reformulation of the leading term of the loading functional (6.7.2), we now have $\underset{\sim}{\tau}^0 = 0$. As a consequence, the convergence (7.1.3) or the estimate (7.1.4) should be replaced by

$$\epsilon\, \|\bar{\underset{\approx}{\rho}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\sim}{\tau}}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \lesssim \epsilon^\theta \tag{7.5.1}$$

or

$$\epsilon\, \|\bar{\underset{\approx}{\rho}}^\epsilon\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\approx}{\gamma}}^\epsilon - \underset{\approx}{\gamma}^0\|_{\underset{\approx}{L}_2^{\mathrm{sym}}(\omega)} + \|\bar{\underset{\sim}{\tau}}^\epsilon\|_{\underset{\sim}{L}_2(\omega)} \to 0 \quad (\epsilon \to 0). \tag{7.5.2}$$

These estimates are similar to (7.3.4) and (7.3.5). Therefore, under the new loading assumption, when the Naghdi model converges, the Koiter also converges. This is the reason why under this new loading assumption, there is no significant difference between the Naghdi-type model and the Koiter-type model.

All the other issues that we discussed in the last few sections can be likewisely discussed under the new loading assumption. In the literature, the classical models are usually defined under the loading assumption of this section.

7.6 Concluding remarks

The model was completely justified for plane strain cylindrical shells, flexural shells, and totally clamped elliptic shells without imposing extra conditions on the shell data. For other membrane–shear shells, the model was only justified under the assumption that the representation $\zeta_*^0 \in W^*$ of the leading term of the resultant loading functional is in the smaller space V^* and the applied forces on the shell is admissible. A rigorous analysis for the case of ζ_*^0 does not belong to V^* is completely lacking. By increasing the number of trial functions in the variational methods, more complicated models can be derived. It seems that the more involved models might be more accurate [42] and the range of applicability might also be enlarged.

The mathematical analysis of the derived model given in this thesis is sufficient for our purpose of proving the model convergence in the relative energy norm, but far from enough for other purposes. For example, for numerical analysis of the Reissner–Mindlin plate model, stronger estimates on the model solution are needed and were established in [7].

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V — closure of W, 47, 66, 133, 176

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Γ^{*k}_{ij} — Christoffel symbol, 99

$\Gamma^{\gamma}_{\alpha\beta}$ — Christoffel symbol on middle surface, 98

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